Al-Nahrain University

Collage of Science

Department of Mathematics and Computer Applications

Real Analysis I /Year Three

Dr. Aamena Al-Qabani

Chapter One ⇒ Real Numbers.

Chapter Two ⇒ Sequence of Real Numbers.

Chapter Three ⇒ Metric Spaces.

Chapter Four ⇒ Continuity.

BOOKS:

1. Principle of Mathematical Analysis, by W. Rudin.

2. Introduction to Real Analysis, by J. DePree and C. Swartz.

3. Introduction to Real Analysis, by A. G. Naoum.

Contents

1 Real Numbers **2**

1.1 Preliminaries 2

1.2 Fundamental Properties of the Real Numbers 5

1.2.1 The Field Axioms 5

1.2.2 The Ordered Set 5

1.2.3 The Completeness Property 7

1.2.4 The Archimedean Principle 8

1.2.5 The Density of Rational Numbers and Irrational Numbers 9

1.2.6 The Absolute Value of a Number 11

2 Sequences of Real Numbers **15**

2.1 Introducing the main concepts 15

2.2 Convergence and Divergence of Sequences 16

2.3 Bounded Sequences and Monotonic Sequences 21

2.3.1 Bounded Sequences 21

2.3.2 Monotonic Sequences 27

2.4 Subsequence 29

2.5 Cauchy Sequences 31

2.6 Countable and Uncountable sets 33

3 Metric Spaces **36**

3.1 Introduction 36

3.2 Basic Principles of Topology 40

3.3 Convergence in Metric Spaces 48
3.4 Compact Metric Spaces 54

4 Continuity **60**
4.1 Definitions and Concepts 60
4.2 Real Valued Functions-The space $C(X)$ 66
4.2.1 Real Continuous Functions Defined on Compact Spaces 67
4.3 The Uniform Continuity 69

Chapter 1

Real Numbers

1.1 Preliminaries

1. The set $\mathbb{N}$ of natural numbers is the set of "counting numbers", simply put

$$\mathbb{N} = \{1, 2, 3, \cdots\} \text{ and } \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}.$$

2. The set $\mathbb{Z}$ is the set of integer numbers,

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}.$$

3. A rational number is a number that can be written as an integer divided by a natural number and the set of rational numbers can be written

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, \cdots \right\}.$$

Some numbers with which we are familiar are not included in the sets mentioned above. For example we will show that $\sqrt{2} \notin \mathbb{Q}$. Also e (the number, approximately equal to 2.71828, used in the exponential function) and our old friend π (approximately equal to 3.14159) are other examples of irrational numbers $\mathbb{Q}'$ not contained in $\mathbb{Q}$. The set $\mathbb{R}$ is the set of real numbers. These are all the numbers in $\mathbb{Q}$ and all of the numbers in between, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

We often picture the set R as a line (the "real line")

The symbol $\subset$ means"is a subset of", so the sets discussed in this page are such that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Example 1.1.1. Show that $x = \sqrt{2}$ is irrational number, that is the equation $x^2 = 2$ has no roots in the rational number set.

Suppose that, there is $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that

$$\sqrt{2} = \frac{p}{q} \Rightarrow p^2 = 2q^2, \tag{1.1}$$

where p and q have no common factors. Therefore

If both p and q are odd numbers then p^2, q^2 are also odd, which is a contradiction, since $2q^2$ is even.

If q is odd number and p is even ($p = 2n, n \in \mathbb{N}$), then q^2 is odd and $p^2 = 4n^2$ is even. Using (1.1) we get $q^2 = 2n^2$ which is a contradiction, since q^2 is odd.

If p is odd number and q is even, we will also get a contradiction. Thus $\sqrt{2} \notin \mathbb{Q}$.

Example 1.1.2. Show that $x = \sqrt{3}$ is irrational number, that is the equation $x^2 = 3$ has no roots in the rational numbers set.

Solution:

Suppose that, there is $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that

$$\sqrt{3} = \frac{p}{q} \Rightarrow p^2 = 3q^2, \tag{1.2}$$

where p and q have no common factors. Therefore p, q both can not be even.

Suppose that both p and q are odd numbers, then there exists $n, m \in \mathbb{N}$ such that

$$p = 2n + 1, \qquad q = 2m + 1.$$

Then

$$(2n+1)^2 = 3(2m+1)^2 \Rightarrow 4n^2 + 4n + 1 = 3(4m^2 + 4m + 1) \Rightarrow 4n^2 + 4n + 1 = 12m^2 + 12m + 3,$$

hence

$$4n^2 + 4n = 12m^2 + 12m + 2 \Rightarrow 2n^2 + 2n = 6m^2 + 6m + 1,$$

which a contradiction, since the left side $(2n^2+2n)$ is even number however the right side $(6m^2+6m+1)$ is odd number.

Also, if p even and q odd then there exists $n, m \in \mathbb{N}$ such that

$$p = 2n, \qquad q = 2m + 1.$$

Then

$$(2n)^2 = 3(2m+1)^2 \Rightarrow 4n^2 = 3(4m^2 + 4m + 1) \Rightarrow 4n^2 = 12m^2 + 12m + 3,$$

which a contradiction, since the left side $(4n^2)$ is even number however the right side $(12m^2 + 12m + 3)$ is odd number.

If p is odd number and q is even, we will also get a contradiction. Thus from all the above cases $\sqrt{3} \notin \mathbb{Q}$.

1.2 Fundamental Properties of the Real Numbers

1.2.1 The Field Axioms

For all $x, y, z \in \mathbb{R}$ we have

1. $x + y \in \mathbb{R}$.
2. $x + y = y + x$.
3. $(x + y) + z = x + (y + z)$.
4. There exists $0 \in \mathbb{R}$ such that $x + 0 = x$ for all $x \in \mathbb{R}$.
5. For each $x \in \mathbb{R}$ there exists $-x \in \mathbb{R}$ such that $x + (-x) = 0$.
6. $xy \in \mathbb{R}$.
7. $xy = yx$.
8. $x(yz) = (xy)z$.
9. There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1x = x$ for all $x \in \mathbb{R}$.
10. For $x \neq 0$ there exists $x^{-1} \in \mathbb{R}$ such that $x(x^{-1}) = 1$.
11. $x(y + z) = xy + xz$.

It is clear that the set of rational numbers $\mathbb{Q}$ with addition and multiplication forms a field, however the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}^{'}$ are not.

1.2.2 The Ordered Set

Definition 1.2.1. Let S be a non empty set. An *order* on S is a relation, denoted by " $<$ ", with the following properties:

i. If $x, y \in S$, then either $x < y$ or $y < x$.

ii. For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

The notation $x \leq y$ means that $x < y$ or $x = y$. In other words, $x \leq y$ is the negation of $x > y$.

Definition 1.2.2. A non empty set S with an order relation is called an *ordered* set.

It is clear that $\mathbb{R}$ is an ordered field.

Definition 1.2.3. Let S be an ordered set and $E \subset S$.

i. If there exists $\alpha \in S$ such that $x \leq \alpha$ for all $x \in E$, then E is *bounded above* and α is called an *upper bound* of E.

ii. If there exists $\beta \in S$ such that $\beta \leq x$ for all $x \in E$, then E is *bounded below* and β is called a *lower bound* of E.

Example 1.2.4. Show which of the following sets are bounded above or bounded below and find the upper and lower bounds if they exist.

1. The set of all natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$.
2. $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.
3. $S = (0, 1)$.
4. $B = [-2, 3)$.
5. $C = \{-2, 0, 2, 4, 6, 8\}$.

Definition 1.2.5. Let S be an ordered set and $E \subset S$. Suppose that there exists $\alpha \in S$ such that

i. α is an upper bound of E, i.e. E is bounded above.

ii. If α_1 is another upper bound of E then $\alpha \leq \alpha_1$.

Then α is called the *least upper bound* of E or α is the *supremum* of E

$$\alpha = \sup E.$$

Definition 1.2.6. Let S be an ordered set and $E \subset S$. Suppose that there exists $\beta \in S$ such that

i. β is a lower bound of E, i.e. E is bounded below.

ii. If β_1 is another lower bound of E then $\beta_1 \leq \beta$.

Then β is called the *greatest lower bound* of E or β is the *infimum* of E

$$\beta = \inf E.$$

Example 1.2.7. Let $E = (1, 2)$. Show that E is bounded below and bounded above and find sup E and inf E.

It is clear that $x < 3$ for all $x \in E$, thus E is bounded above and 3 is an upper bound for E. Also $0 < x$ for all $x \in E$, thus E is bounded below and 0 is a lower bound for E.
It is not difficult to see that $2 = \sup E$ and $1 = \inf E$.

Example 1.2.8. Let $E = (-1, 3]$. Show that E is bounded below and bounded above and find sup E and inf E.

- - - -

The Completeness Axiom

The set of all real numbers $\mathbb{R}$ is *complete*, that is every non empty subset of $\mathbb{R}$ that is bounded above has a supremum. Therefore $\mathbb{R}$ is a **complete ordered Field**.

Example 1.2.11. Show that the set of all integer numbers $\mathbb{Z}$ has the least upper bound property.

Solution: Let S be a non-empty subset $\mathbb{Z}$, (the set of all integer numbers) such that S is bounded above. This means that there exists $a \in \mathbb{Z}$ such that $a \geq x$ for all $x \in S$.

Let L be the set of all upper bounds of S, that is $L \subseteq \mathbb{Z}$. Since L is an ordered set then there exists $n \in$ Ł such that $n < a$ for all $a \in$ Ł. This means that n is an upper bound of S and it is the smallest one. Thus n is the lest upper bound of S.

Therefore $\mathbb{Z}$ is complete set.

Example 1.2.12. Show that the Field of rational numbers $\mathbb{Q}$ is incomplete.

- - - - - - - - - - - - - - - - - Give an example of a set in $\mathbb{Q}$ which is bounded below or bounded above and show that it has no inf or sup- -

1.2.4 The Archimedean Principle

Theorem 1.2.13. *Let $x, y \in \mathbb{R}$. If $x > 0$ then there exists a positive integer n such that*

$$nx > y.$$

Proof. Suppose that $A = \{nx : n \in \mathbb{N}\}$, then $A \subset \mathbb{R}$. If the theorem is not true then y will be an upper bound for A. By the completeness axiom of $\mathbb{R}$, A has a least upper bound say α. That is

$$\alpha = \sup A.$$

Since $x > 0$, then $\alpha - x < \alpha$ and $\alpha - x$ is not upper bound of A. Hence there exists $m \in \mathbb{N}$ such that $\alpha - x < mx$. This implies that

$$\alpha < mx + x,$$

that is

$$\alpha < x(m + 1)$$

which is a contradiction since $\alpha = \sup A$. Therefore the theorem is true. □

Corollary 1.2.14. *Let ε be a positive real number. Then there exists a positive integer n such that*

$$\frac{1}{n} < \varepsilon.$$

Proof. - - -

Proof. - - -

Remark 1.2.19. You might be wondering is the completeness axiom of $\mathbb{R}$ relates to the existence of infimum for subsets of $\mathbb{R}$ bounded from below? In fact, the completeness axiom is equivalent to the next statement:

Every subset of $\mathbb{R}$ which is bounded from below, has infimum.

1.2.6 The Absolute Value of a Number

When dealing with inequalities and with negative numbers, some times all we care about is the "magnitude" of the number, rather that its "sign". The "magnitude" soon to be coined "absolute value", is essentially the *distance of the number from zero*. For example, both 5 and -5 have distance from the number 0 equal to (non-negative number) 5. To end this we have the next concept.

Definition 1.2.20. Let $x \in \mathbb{R}$ be any real number. We define the *absolute value of the number* x, typically symbolised as "$|x|$", to the next *non-negative* number:

$$|x| := \begin{cases} x, & \text{when } x > 0, \\ 0, & \text{when } x = 0, \\ -x & \text{when } x < 0. \end{cases}$$

The number $|x|$ sometimes is also called the "modulus" of x. For example $|3| = 3$, $|0| = 0$, $|-7| = 7$. The function $|\cdot| : \mathbb{R} \to \mathbb{R}^+$ is called the *Absolute Value Function*.

The next result lists some of the main properties of the absolute value function.

Theorem 1.2.21. *Let $x, y \in \mathbb{R}$. Then the following are true:*

i. $|xy| = |x||y|$.

ii. $|x|^2 = x^2$.

iii. If $k \geq 0$, *then we have*

$$|x| \leq k \iff -k \leq x \leq k.$$

iv. $-|x| \leq x \leq |x|$.

** The symbol "$\iff$" means "equivalence" and should be read as "if and only if".*

Proof. **i.**

Let $x, y \in \mathbb{R}$. Then we have four cases:

(1) if $x > 0$ and $y > 0$, then $xy > 0$.

Using the definition of the absolute value we get

$|x| = x$, $|y| = y$ and $|xy| = xy$. Thus $|xy| = |x||y|$.

(2) if $x > 0$ and $y < 0$, then $xy < 0$.

Using the definition of the absolute value we get

$|x| = x$, $|y| = -y$ and $|xy| = -xy$. Thus $|xy| = |x||y|$.

(3) if $x < 0$ and $y > 0$, then $xy < 0$.

Using the definition of the absolute value we get

$|x| = -x$, $|y| = y$ and $|xy| = -xy$. Thus $|xy| = |x||y|$.

(4) if $x < 0$ and $y < 0$, then $xy > 0$.

Using the definition of the absolute value we get

$|x| = -x$, $|y| = -y$ and $|xy| = xy$. Thus $|xy| = |x||y|$.

iii. If $k \geq 0$, *then we have*

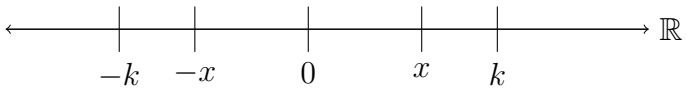
$$|x| \leq k \iff -k \leq x \leq k.$$

Proof. Let $k \geq 0$. Suppose $|x| \leq k$ we will show that $-k \leq x \leq k$.

We have two cases:

(1) if $x > 0$, then $|x| = x$. Hence $x \leq k$ and since $x > 0$ then it is clear that $x > -k$. Thus

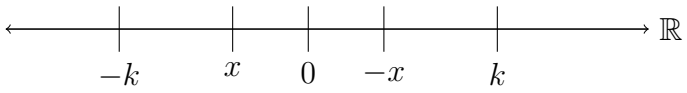
$$-k \leq x \leq k.$$



(2) if $x < 0$, then $|x| = -x$. Hence $-x \leq k \Rightarrow x \geq -k$ and since $x < 0$ then it is clear that $x \leq k$.

Thus

$$-k \leq x \leq k.$$



Now Suppose $-k \leq x \leq k$ we will show that $|x| \leq k$.

We have two cases:

(1) if $x > 0$, then $|x| = x$.

From $-k \leq x \leq k$, we get $x \leq k$, thus $|x| \leq k$.

(2) if $x < 0$, then $|x| = -x$.

From $-k \leq x \leq k$, we get $-k \leq x$, thus $k \geq -x$. This implies that $|x| \leq k$.

□

The following inequality is a very important, yet simple, consequence of the definition of the absolute value.

Corollary 1.2.22 (Triangle inequality)**.** *For any $x, y \in \mathbb{R}$, we have*

$$|x + y| \leq |x| + |y|.$$

Proof. By Theorem 1.2.19(iv), we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. We may add these inequalities to fine

$$-(|x| + |y|) \leq x + y \leq (|x| + |y|).$$

By Theorem 1.2.19(iii), we deduce that $|x + y| \leq |x| + |y|$, as desired. □

Corollary 1.2.23 (More triangle inequalities)**.** *For any $x, y \in \mathbb{R}$, we have*

i. $|x - y| \leq |x| + |y|$.

ii. $||x| - |y|| \leq |x - y|$.

Proof. **i** $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.

Thus $|x - y| \leq |x| + |y|$.

- - - -

Chapter 2

Sequences of Real Numbers

A very important object in Analysis is that of a sequence, as defined below.

2.1 Introducing the main concepts

Definition 2.1.1. A sequence of real numbers or sequence in $\mathbb{R}$ is a function (a correspondence) $f : \mathbb{N} \to \mathbb{R}$, such that, to each number $n \in \mathbb{N}$, associates a unique real number $a_n \in \mathbb{R}$:

$$\mathbb{N} \ni n \mapsto a_n \in \mathbb{R}.$$

Sequences will be denoted by either of the next equivalent notations

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_1, a_2, a_3, \cdots\}.$$

The numbers $a_1, a_2, \cdots$ are called the *terms of the sequence*. The number a_n is called the *nth term of the sequence*.

Example 2.1.2. The following functions define sequences:

1. $a_n := a$, for all $n \in \mathbb{N}$, where $a \in \mathbb{R}$ is a given number.

The terms of this sequence remain constant as n increases: $a, a, a, \cdots$.

2. $a_n := -n$, for all $n \in \mathbb{N}$.

The terms of this sequence increase uncontrollably to $-\infty$ as n increases: $-1, -2, -3, \cdots$.

3. $a_n := \frac{1}{n^2+\sqrt{2}}$, for all $n \in \mathbb{N}$. The terms of this sequence seems to get closer and closer to 0 as n increases: $\frac{1}{1+\sqrt{2}}, \frac{1}{4+\sqrt{2}}, \frac{1}{9+\sqrt{2}}, \frac{1}{16+\sqrt{2}}, \cdots$.

4. $a_n := (-1)^n n^{-7} + 3$, for all $n \in \mathbb{N}$.

The terms of this sequence oscillate but settle closer and closer to 3, whilst having interchanging sign as n increases: $-1+3, 2^{-7}+3, -3^{-7}+3, 4^{-7}+3, \cdots$.

5. $a_n := \cos^2(3^{n+1})$, for all $n \in \mathbb{N}$.

- - - -

and smaller as n increases. Intuitively, $\frac{2^n-1}{2^n}$ tends to 1.

2. The sequence $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$; i.e. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$. All the terms are positive but get progressively closer to zero, indeed the distance of a_n from $a = 0$ becomes smaller and smaller as n increases. Intuitively, $\frac{1}{n}$ tends to 0.

Definition 2.2.2 (Definition of a Limit)**.** Start with a general sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, and a real number a. What the statement "a_n tends to a" means? Well, it means that the distance between a_n and a gets *arbitrarily small*, and that is should happen not just for some value of n, but for all large n; i.e. for all value of n from some point onwards.

In mathematical language, given any band around a no matter how narrow, the sequence a_n will eventually lie in this band. Mathematically speaking this says, given any positive number ε, $a - \varepsilon < a_n < a + \varepsilon$ for all n from some point onwards. Note however, that $a - \varepsilon < a_n < a + \varepsilon$ is the same as $|a_n - a| < \varepsilon$.

Definition 2.2.3 (The convergence)**.** Let a_n be a sequence of real numbers, and let $a \in \mathbb{R}$. We say a_n **converges** to a as $n \to \infty$ if, for any $\varepsilon > 0$, there exists an integer k such that for $n \geq k$

$$|a_n - a| < \varepsilon.$$

We also say a_n **tends** to a, and we write, $a_n \to a$, or

$$\lim_{n\to\infty} a_n = a.$$

As such, a_n is called a **convergent** sequence, and a is the **limit** of a_n.

We say a_n **diverges** if it does not converge.

***** It is important to realise that ε is given first, then an appropriate k is found for which $|a_n - a| < \varepsilon$ for all $n \geq k$.

Example 2.2.4.

1. Let $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that a_n tends to zero as $n \to \infty$.

Proof. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$ (this is possible by Archimedean property). Then for $n \geq k$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{k} < \varepsilon.$$

Hence, by the definition of a limit, $\frac{1}{n} \to 0$ as $n \to \infty$.

2. Prove that $1 - \frac{1}{2^n}$ tends to 1 as $n \to \infty$.

Proof. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$ (this is possible by Archimedean property). Then for $n \geq k$,

$$\left|1 - \frac{1}{2^n} - 1\right| = \frac{1}{2^n} \leq \frac{1}{n} \leq \frac{1}{k} < \varepsilon.$$

Hence, by the definition of a limit, $\frac{1}{2^n} \to 1$ as $n \to \infty$.

3. Let $a_n := \frac{4n+1}{2n-1}$ for all $n \in \mathbb{N}$. Prove that a_n converges to 2 as $n \to \infty$.

Proof. First note that

$$\left|\frac{4n+1}{2n-1} - 2\right| = \left|\frac{4n+1-2(2n-1)}{2n-1}\right| = \frac{3}{2n-1} \leq \frac{3}{n}.$$

Hence, for $\varepsilon > 0$ we can choose $k \in \mathbb{N}$ such that $\frac{3}{k} < \varepsilon$ when $\frac{1}{k} < \frac{3}{\varepsilon}$. Thus for $n \geq k$,

$$\left|\frac{4n+1}{2n-1} - 2\right| \leq \frac{3}{n} \leq \frac{3}{k} < \varepsilon.$$

Therefore by the definition of convergence, $\frac{4n+1}{2n-1} \to 2$ as $n \to \infty$.

Quiz 6: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that

$$a_n = \begin{cases} 0, & \text{when } n \text{ is odd,} \\ 2n, & \text{when } n \text{ is even.} \end{cases}$$

Does a_n converge? Why? Does a_n has a convergent subsequence? Explain.

:Solution:

The sequence is divergent, because it is unbounded. To show this we will use the contradiction.

Suppose that a_n is bounded sequence, that is there exists $M > 0$ such that

$$|a_n| \leq M \quad \text{for all } n \in \mathbb{N},$$

hence

$$|a_n| \leq M \quad \text{for all even } n \in \mathbb{N}.$$

Thus $2n \leq M$ for all even $n \in \mathbb{N}$, this implies, $n \leq \frac{M}{2}$, for all even $n \in \mathbb{N}$. This means that the set of all even positive integer numbers is bounded, Which a contradiction. Thus a_n is unbounded and therefore it is not convergent.

Now , we have $a_n := 0, 2, 0, 4, 0, 6, 0, 8, \cdots$

if we choose $b_{n_k} := 0, 0, 0, 0, \cdots$, where $n \in \mathbb{N}$ odd number and $k = 1, 2, 3, \cdots$, then $\{b_{n_k}\}_{k=1}^{\infty}$ is a subsequence of a_n and $\{b_{n_k}\}_{k=1}^{\infty}$ is a constant sequence that convergent to 0.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that

$$a_n = \begin{cases} n, & \text{when } n \leq 1000, \\ 1, & \text{when } n > 1000. \end{cases}$$

Does a_n converge (has a limit)? why?

Solution. Let $\varepsilon > 0$. Choose $k = 1000$. Then for all $n > k$,

$$|a_n - 1| = 1 - 1 = 0 < \varepsilon.$$

Hence, by the definition of a limit $a_n \to 1$ as $n \to \infty$. Thus a_n converge.

5. Let $a_n := (-1)^n$ for all $n \in \mathbb{N}$. Does a_n has a limit (converges)? why?

Solution.

If we suppose that $(-1)^n \to 1$ as $n \to \infty$ then for $\varepsilon = 0.5$. Then $a_n \in (0.5, 1.5)$ when n is even. That is $0.5 < a_n < 1.5$ for all $n \in \mathbb{N}$ and n is even. But $a_n \notin (0.5, 1.5)$ when n is odd. In particular there are an infinite number of terms which are not in the interval $(0.5, 1.5)$. Hence $a_n \nrightarrow 1$.

Also, if we suppose that $(-1)^n \to -1$ as $n \to \infty$ then $a_n \in (-1.5, -0.5)$ when n is odd. That is $-1.5 < a_n < -0.5$ for all $n \in \mathbb{N}$ and n is odd. But $a_n \notin (-1.5, -0.5)$ when n is even. In particular there are an infinite number of terms which are not in the interval $(-1.5, -0.5)$. Hence $a_n \nrightarrow -1$.

Now suppose that there exists $a_0 \in \mathbb{R}$ with $a \neq 1$ and $a \neq -1$. Let $|-1-a_0| = \varepsilon_1 > 0$ and $|1-a_0| = \varepsilon_2 > 0$. If we choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon_1$ and $\varepsilon < \varepsilon_2$ then we note that $a_n \notin (a_0-\varepsilon, a_0+\varepsilon)$ for all $n \in \mathbb{N}$. Thus $a_n \nrightarrow a_0$. Therefore a_n does not converge.

We have already claimed that not all sequences converge. How do we show that a sequence fails to converge?

Definition 2.2.5 (Negation of convergence)**.**
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ and $a_0 \in \mathbb{R}$. Then $\{a_n\}_{n=1}^{\infty}$ does not converge to a_0 as $n \to \infty$ when:
Exists $\varepsilon_0 > 0$ **such that for any** $k \in \mathbb{N}$**, there exists** $n \in \mathbb{N}$ **with** $n \geq k$ **and** $|a_n - a_0| \geq \varepsilon_0$.

The next result shows that, if the limit of a sequence exists, it is unique.

Theorem 2.2.6 (Uniqueness of Limits)**.**
Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ *of real numbers converges to both* a_1 *and* a_2*, where* a_1,$a_2 \in \mathbb{R}$*:*

$$\lim_{n\to\infty} a_n = a_1 \qquad \textit{and} \qquad \lim_{n\to\infty} a_n = a_2.$$

Then, $a_1 = a_2$*. Hence, every sequence has at most one limit.*

Proof. Suppose for a contradiction, that $a_1 \neq a_2$. Let $\varepsilon = |a_1 - a_2| > 0$.
By definition of a limit, for any $\frac{\varepsilon}{2} > 0$ there exists $k_1, k_2 \in \mathbb{N}$ such that

$$|a_n - a_1| < \frac{\varepsilon}{2} \text{ for all } n \geq k_1$$

$$|a_n - a_2| < \frac{\varepsilon}{2} \text{ for all } n \geq k_2$$

Now we fix $k \in \mathbb{N}$ greater than both k_1, k_2:

$$k \geq \max\{k_1, k_2\}.$$

Then, for any such k, the Triangle Inequality gives

$$\varepsilon = |a_1 - a_2| = |a_1 - a_k + a_k - a_2|$$

$$\leq |a_1 - a_k| + |a_k - a_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

that is $\varepsilon < \varepsilon$ which is a contradiction. Hence, we have obtained that $a_1 = a_2$, as desired. □

2.3 Bounded Sequences and Monotonic Sequences

Showing that a sequence does not converge may not be always easy. However, there is an easier condition one can use to show that a sequence can not converge to any real number. In this section we will study some necessary and sufficient conditions for convergence of sequences.

2.3.1 Bounded Sequences

Definition 2.3.1. The set A is called **bounded** if it bounded both from above and from below.
A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called **bounded** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded. Or, in other words, there exists M_1 and M_2 such that

$$M_1 \leq a_n \leq M_2 \quad \text{for all } n \in \mathbb{N}.$$

Or, there exists $M > 0$ such that

$$|a_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Proposition 2.3.2 (Convergent sequences are bounded)**.**

Every convergent sequence is bounded.

Proof. Suppose that $a_n \to a$ as $n \to \infty$. Let us choose $\varepsilon = 1 > 0$ in the definition of a limit, there

exists $k \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq k$, that is

$$a - 1 < a_n < a + 1 \text{ for all } n \geq k.$$

Hence, by the triangle inequality, we have

$$|a_n| < |a| + 1 \text{ for all } n \geq k.$$

But for $n = 1, 2, \cdots, k-1$, a_n is necessarily bounded as $\{a_1, a_2, \cdots, a_{k-1}\}$ is a finite set. Thus if we set

$$M = max\{|a_1|, |a_2|, \cdots, |a_{k-1}|, |a| + 1\}$$

which is a positive number, then $|a_n| \leq M$ for all $n \in \mathbb{N}$, and $\{a_n\}$ is bounded. □

Remark 2.3.3.

i. The contrapositive of this is the statement: **if a_n is unbounded, then a_n diverges**. This can be useful to show that a sequence does not converge. For example, the sequence $\sqrt{n}$ is unbounded, and hence divergent. To prove unboundedness, argue by contradiction: if $\sqrt{n} < M$ for all n (for some M), then $n < M^2$ for all n, which implies that $\mathbb{N}$ is bounded − a contradiction. Hence $\sqrt{n}$ is unbounded.

ii. The converse is not true however. There are sequences which are bounded but do not converge. For example $(-1)^n$ is clearly bounded (by -1 and 1) but, as we have noticed, $(-1)^n$ diverges.

Rules for limits

It is clearly rather time consuming (and inefficient) to prove that a given sequence converges to a given limit by going back to the definition. This is particularly wasteful if the sequence we are interested is similar to sequences whose limit we already know. For example, suppose we wanted to know the limiting behaviours of the sequence

$$a_n = \frac{1}{n} + \left(1 - \frac{1}{2^n}\right).$$

Since we know that $\frac{1}{n} \to 0$ and $1 - \frac{1}{2^n} \to 1$ it seems intuitively clear that a_n should tend to 1.

Theorem 2.3.4 (Rules for limits)**.**

Suppose that a_n and b_n are sequences such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Then the following hold:

i. $a_n + b_n \to a + b$ (*the* sum *rule*);

ii. $a_n b_n \to ab$ (*the* product *rule*);

iii. if $b_n \neq 0$ for all n and $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$ (*the* quotient *rule*);

iv. $|a_n| \to |a|$;

As a special case of (b) we have $a_n \to a \Longrightarrow \lambda a_n \to \lambda a$.

Suppose that a_n be a sequence such that $a_n \to a$ as $n \to \infty$. Prove that $|a_n| \to |a|$. Does that converse true? Why? Explain your answer.

Proof. **(i)** Let $\varepsilon > 0$. Since $a_n \to a$ and $b_n \to b$, then there exists $k_1, k_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \quad \text{for all } n \geq k_1$$

$$|b_n - b| < \frac{\varepsilon}{2} \quad \text{for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$. Then for all $n \geq k$

$$|a_n - a| < \frac{\varepsilon}{2} \quad \text{for all } n \geq k$$

$$|b_n - b| < \frac{\varepsilon}{2} \quad \text{for all } n \geq k.$$

Hence for all $n > k$,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It follows that $a_n + b_n \to a + b$ as $n \to \infty$.

Proof. **(ii)** First we note that

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab| = |b_n||a_n - a| + |a||b_n - b|.$$

But a_n and b_n are convergent sequences and hence bounded, so there exists $M_1 > 0$ and $M_2 > 0$ such that $|a_n| < M_1$ for all n with $|a| < M_1$ and $|b_n| < M_2$ for all n with $|b| < M_2$. Thus

$$|a_n b_n - ab| \leq M_1|a_n - a| + M_2|b_n - b|. \tag{2.1}$$

Now, let $\varepsilon > 0$, there exists positive integers k_1 and k_2 such that

$$|a_n - a| < \frac{\varepsilon}{2M_1} \quad \text{for all } n > k_1.$$

$$|b_n - b| < \frac{\varepsilon}{2M_2} \quad \text{for all } n > k_2.$$

Let $k = \max\{k_1, k_2\}$. Then for all $n > k$, the above both hold and hence (2.1) gives

$$|a_n b_n - ab| \leq M_1 \frac{\varepsilon}{2M_1} + M_2 \frac{\varepsilon}{2M_2} < \varepsilon.$$

It follows that $a_n b_n \to ab$ as $n \to \infty$.

Proof. **(iii)** It sufficient to show that $\frac{1}{b_n} \to \frac{1}{b}$, because we can use the formula we have already proved for the product. First we note that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{1}{|b||b_n|}\,|b_n - b|$$

As $b_n \to b$, we have for any $\varepsilon > 0$, that $|b_n - b| < \varepsilon$ for all n sufficiently large. Take $\varepsilon < \frac{|b|}{2}$. Thus there exists $k_1 \in \mathbb{N}$ such that for all $n > k_1$

$$|b_n| = |b - (b - b_n)| \geq |b| - |b - b_n| > \frac{|b|}{2},$$

hence

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b||b_n|}\,|b_n - b| \leq \frac{2}{|b|^2}|b_n - b|.$$

Also, since $b_n \to b$ then for each $\varepsilon > 0$ (take $\varepsilon = \frac{\varepsilon|b|^2}{2}$), there exists $k_2 \in \mathbb{N}$. Hence, for $k = \max k_1, k_2$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon \quad \text{for all } n > k.$$

□

Example 2.3.5.

1. Show that

$$\lim_{n\to\infty} \left\{\frac{1}{n} + \left(1 - \frac{1}{2^n}\right)\right\} = 1.$$

We already know that $\frac{1}{n} \to 0$ and $1 - \frac{1}{2^n} \to 1$ as $n \to \infty$. Hence by the sum rule, $\frac{1}{n} + \left(1 - \frac{1}{2^n}\right) \to 0 + 1 = 1$.

2. Show that

$$\lim_{n\to\infty} \frac{n+2}{n} = 1.$$

We have that $\frac{n+2}{n} = 1 + \frac{2}{n}$. But from the product rule, $\frac{2}{n} = 2\frac{1}{n} \to 0$, so, by the sum rule,

$$\frac{n+2}{n} = 1 + \frac{2}{n} = 1 = 0 = 1.$$

3. Show that

$$\lim_{n\to\infty} \frac{2n^2+1}{3n^2+n} = \frac{2}{3}.$$

We have

$$\frac{2+\frac{1}{n^2}}{3+\frac{1}{n}}.$$

The numerator is $2 + \frac{1}{n}.\frac{1}{n} \to 2 + 0.0 = 2$ by the sum and the product rules, while the denominator is $3 + \frac{1}{n} \to 3$. Thus by the quotient rule,

$$\frac{2n^2+1}{3n^2+n} \to \frac{2}{3}.$$

Theorem 2.3.6 (The Sandwich Rule)**.** *Suppose that a_n, b_n and c_n are sequences such that*

$$a_n \leq b_n \leq c_n \quad \textit{for all } n.$$

Further suppose that $a_n \to a$ and $c_n \to a$. Then $b_n \to a$.

The sequence b_n is stuck between a_n and b_n. Both a_n and c_n. Both a_n and c_n tend to a and b_n is squeezed to a.

Proof. Let $\varepsilon > 0$, there exists $k_1, k_2 \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon \text{ for all } n \geq k_1 \Rightarrow a - \varepsilon < a_n < a + \varepsilon \text{ for all } n \geq k_1$$

$$|b_n - a| < \varepsilon \text{ for all } n \geq k_2 \Rightarrow a - \varepsilon < b_n < a + \varepsilon \text{ for all } n \geq k_2$$

Let $k = max\{k_1, k_2\}$. Thus for $n \geq k$, both the above hold, that is

$$|a_n - a| < \varepsilon \text{ for all } n \geq k \Rightarrow a - \varepsilon < a_n < a + \varepsilon \text{ for all } n \geq k$$

$$|c_n - a| < \varepsilon \text{ for all } n \geq k \Rightarrow a - \varepsilon < c_n < a + \varepsilon \text{ for all } n \geq k.$$

Since $a_n \leq b_n \leq c_n$ then

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon \text{ for all } n \geq k.$$

That is

$$a - \varepsilon < b_n < a + \varepsilon \text{ for all } n \geq k \Rightarrow \text{i.e.} \Rightarrow |b_n - a| < \varepsilon \text{ for all } n > k.$$

Hence

$$b_n \to a.$$

□

Example 2.3.7. *Show that* $\frac{(-1)^n}{n} \to 0$.

Note that

$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}.$$

But $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, so by the Sandwich rule, $\frac{(-1)^n}{n} \to 0$ also.

Proposition 2.3.8.

i. *Suppose that* $a_n \to a$ *as* $n \to \infty$ *and* $a_n \geq 0$ *for all* n. *Then* $a \geq 0$.

ii. *Suppose that* $a_n \to a$ *and* $b_n \to b$ *and that* $a_n \geq b_n$ *for all* n. *Then* $a \geq b$.

Remark This is **NOT** true if each $\geq$ is replaced by $>$. For example, $\frac{1}{n} > 0$ for all n, but the limit (0) is not > 0.

Some Fundamental Limits Now that we have establish a number of rules and properties of limits, it is useful to know the limits of number of frequently occurring sequences. With these, many other limits can be worked out.

Theorem 2.3.9 (Some basic limits)**.**

Let $a > 0$ *and* q *a positive rational number. Then we have the following limits (all as* $n \to \infty$*):*

i. $\frac{1}{n^q} \to 0$;

ii. $a^n \to 0$ if $a < 1$; a^n is unbounded if $a > 1$;

iii. $\sqrt[n]{a} \to 1$;

iv. $\sqrt[n]{n} \to 1$;

v. $\frac{n^q}{a^n} \to 0$ if $a > 1$; $n^q a^n \to 0$ if $a < 1$;

vi. $\frac{p(n)}{a^n} \to 0$ if $a > 1$, for any polynomial $p(n)$;

vii. $\frac{a^n}{n!} \to 0$.

Example 2.3.10.

(1) Show that $\sqrt[n]{\frac{2}{n^2}} \to 1$ as $n \to \infty$. We have

$$\sqrt[n]{\frac{2}{n^2}} = \frac{\sqrt[n]{2}}{\sqrt[n]{n^2}} = \frac{\sqrt[n]{2}}{(\sqrt[n]{n})^2}.$$

By (iii) and (iv), $\sqrt[n]{2} \to 1$ and $\sqrt[n]{n} \to 1$, so also $(\sqrt[n]{n})^2 \to 1$. By the quotient rule, $\frac{\sqrt[n]{2}}{(\sqrt[n]{n})^2} \to 1$.

(2) Show that $\frac{2^n+5n}{3^n+n^2} \to 0$.

Dividing top and bottom of the fraction by the largest term (i.e. 3^n) gives

$$\frac{2^n + 5n}{3^n + n^2} = \frac{(\frac{2}{3})^n + \frac{5n}{3^n}}{1 + \frac{n^2}{3^n}}.$$

The numerator tends to 0 and the denominator to 1 (using (ii) and (v)). By the quotient rule, the limit is 0.

(3) Show that $\frac{n!+(-1)^n}{2^n+(3)n!} \to \frac{1}{3}$.

- - - -

Definition 2.3.11. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then $\{a_n\}_{n=1}^{\infty}$ is called **increasing sequence** if

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots, \quad \text{for all } n \in \mathbb{N},$$

and it is called **decreasing sequence** if

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots, \quad \text{for all } n \in \mathbb{N}.$$

We say that $\{a_n\}_{n=1}^{\infty}$ is **monotone** (or $\{a_n\}_{n=1}^{\infty}$ is a **monotonic sequence**) if it is either increasing or decreasing.

Note that a constant sequence is both increasing and decreasing.

We now come to a fundamental theorem regarding monotone sequences.

Theorem 2.3.12 (Bounded monotonic sequences converge)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence in $\mathbb{R}$. Then*

$$\{a_n\}_{n=1}^{\infty} \text{ is bounded } \iff \{a_n\}_{n=1}^{\infty} \text{ is convergent.}$$

Moreover, if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and increasing, then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

If the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and decreasing, then

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Proof.
Suppose that $\{a_n\}_{n=1}^{\infty}$ is convergent. We have already proved in that every convergent sequence is bounded(and in fact it is not even needed to be monotone for this).

Conversely, suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded and increasing. Then there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then by the completeness of $\mathbb{R}$ the supremum of the set $A := \{a_n : n \in \mathbb{N}\}$ exists. Let us set

$$a := \sup A.$$

We will show that $a_n \to a$ as $n \to \infty$. By the characterisation of the supremum, for any $\varepsilon > 0$ there

exists $a_k \in A$ such that

$$a - \varepsilon \leq a_k.$$

Since $\{a_n\}_{n=1}^{\infty}$ is increasing, we have $a_n \geq a_k$ for all $n \geq k$. Since a is the supremum, it is an upper bound too and hence $a_n \leq a$ for all $n \in \mathbb{N}$. Thus we have shown that

$$a - \varepsilon \leq a_n \leq a, \quad \text{for all } n \geq k,$$

which is equivalent to $-\varepsilon \leq a_n - a \leq 0$ for all $n \geq k$. By the definition of the absolute value and of the limit, we have obtained $\lim_{n\to\infty} a_n = a$.

The case of the decreasing sequence involving the infimum is essentially identical and can be proved by arguing symmetrically (**the details are left as an exercise**). □

Example 2.3.13.
Let a_n be defined by $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. Show that a_n is increasing and that $\lim_{n\to\infty} a_n = 2$.

a_n **is increasing:** $a_2 = \sqrt{2} > a_1$. Now suppose that $a_{n+1} \geq a_n$ for some $n \geq 1$. Then $a_{n+2} = \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} = a_{n+1}$. By mathematical induction $a_{n+1} \geq a_n$ for all n.

a_n **is bounded:** we show $a_n \leq 2$ for all n. $a_1 = 0 < 2$. Now suppose that $a_n \leq 2$ for some $n \geq 1$. Then

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} = \sqrt{4} = 2.$$

Thus by induction we have that $a_n \leq 2$ for all n. In particular $|a_n| \leq 2$ for all n.

Hence a_n is increasing and bounded, and by the previous theorem a_n tends to a limit, say a. That is $a_n \to a$. But then $a_n^2 \to a^2$ that is $2 + a_n \to a^2$ and so, $a^2 = 2 + a$ (this is because $a_n \to a$ and $2 + a_n \to 2 + a$, since the limit is unique, thus $a^2 = 2 + a$). Factorising gives $(a - 2)(a + 1) = 0$; i.e. $a = 2$ or $a = -1$. But $a_n \geq 0$, so $a = -1$ is impossible. Thus $a_n \to 2$.

2.4 Subsequence

Consider a sequence $a_1, a_2, a_3, \cdots$ of real numbers. If we choose every second term we obtain a new sequence $a_2, a_4, a_6, \cdots$. Similarly if we start at a_1 and choose every third term we obtain a new sequence $a_1, a_4, a_7, \cdots$. Or, taking each a_n where n is a square. We obtain the sequence $a_1, a_4, a_9, a_{16}, \cdots$. In each case we obtain a new sequence made up from the terms of the original sequence. Such sequences

are called **Subsequences**.

More formally we have:

Definition 2.4.1. Let $\{a_n\}_{n=1}^{n}$ be a sequence of real numbers, and let n_k ($k \in \mathbb{N}$) be a strictly increasing sequence of natural numbers. Then the sequence $b_k = a_{n_k}$ is a **subsequence** of $\{a_n\}_{n=1}^{n}$.

Proposition 2.4.2.

(1) *Suppose that a_n converges to a. Then every subsequence of a_n converges to a.*

(2) *Every sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ has a monotonic subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, namely a subsequence which is either increasing or decreasing.*

The **main** result in this section is

Theorem 2.4.3 (Bolzano-Weierstrass Theorem)**.**

*Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Then $\{a_n\}_{n=1}^{\infty}$ has a **convergent subsequence** $\{a_{n_k}\}_{k=1}^{\infty}$. Namely:*

$$\textit{Exists } a \in \mathbb{R} \textit{ and } \{a_{n_k}\}_{k=1}^{\infty} \subseteq \{a_n\}_{n=1}^{\infty} : \lim_{k\to\infty} a_{n_k} = a.$$

Remark 2.4.4. In general, bounded sequences may not converge fully, so the result can not be improved to show that the entire sequence converges. For example, the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = (-1)^n$ we have the subsequences

$$a_{n_k} := (-1)^{2k} = 1 \quad \text{and } a_{m_k} := (-1)^{2k+1} = -1$$

both converge to $+1$ and -1 respectively, but the full sequence $\{a_n\}_{n=1}^{\infty}$ does not converge.

Proof. Theorem 2.4.3

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. The by Proposition 2.4.2, it has a monotonic subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. By Theorem 2.3.12, this subsequence is convergent. □

Example 2.4.5. *Suppose that we are asked to check whether the sequence $\{a_n\}_{n=1}^{\infty}$ or any of its subsequence converges, where a_n is given by*

$$a_n := cos^n\left(3^n - \ln n + \sqrt{5}\right), \quad n \in \mathbb{N}.$$

We have

$cos(3 - \ln 1 + \sqrt{5}), cos^2(3^2 - \ln 2 + \sqrt{5}), cos^3(3^3 - \ln 3 + \sqrt{5}), \cdots$

We know that, $-1 \leq \cos x \leq 1$, for all $x \in \mathbb{R}$. Thus $|\cos x| \leq 1$, for all $x \in \mathbb{R}$.

Hence it is clear that $|\cos(3^n - \ln n + \sqrt{5})|^n \leq 1$, as $n \to \infty$.

This means that this sequence is bounded, therefore every subsequence of this sequence is converge.

2.5 Cauchy Sequences

A central problem when trying to check whether a sequence is convergent is that we must a priori know the candidate limit value, in order to apply the definition. Although we have already given examples of how one might be able to guess the limit, this is not always easy. However, there is a condition is very useful in that it allows to check whether a sequence is convergent, without the need to know the putative limit! This is given in the next definition.

Definition 2.5.1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and. We say that the **sequence** $\{a_n\}_{n=1}^{\infty}$ **is Cauchy** when:
For any $\varepsilon > 0$ **(as small as we like), there exists an** $k \in \mathbb{N}$ **(depending on** ε**), such that**

$$|a_n - a_m| < \varepsilon, \quad \text{for all } n, m \geq k.$$

The idea of the concept is that *" as n, m increase, the distance of any two terms $|a_n - a_m|$ decreases, if we look at large enough indices "*. We begin by showing the next result which is very similar to Proposition 2.3.2 for convergent sequences.

Proposition 2.5.2 (Cauchy sequence are bounded)**.**

Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then, the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded. Namely, there exists $M > 0$ such that

$$|a_n| \leq M, \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Let us choose $\varepsilon := 1 > 0$. Since the sequence is Cauchy, there exists $k \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq k$. Hence, by the triangle inequality, for $m := k$, we have

$$|a_n| \leq |a_k| + 1, \quad \text{for all } n \geq k.$$

Finally, we set

$$M := max\,\{|a_1|, |a_2|, \cdots, |a_{k-1}|, |a_k| + 1\}$$

which is a positive number, to find that $|a_n| \leq M$ for all $n \in \mathbb{N}$. □

Theorem 2.5.3 (Convergent $\Longleftrightarrow$ Cauchy)**.**
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then $\{a_n\}_{n=1}^{\infty}$ is Cauchy if and only if it is convergent (to some $a \in \mathbb{R}$).

Proof. ($\Longrightarrow$) Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$ with $a_n \to a$. Then for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $n \geq k$

$$|a_n - a| < \frac{\varepsilon}{2}.$$

Thus for $n, m \geq k$,

$$|a_n - a_m| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
Conversely ($\Longleftarrow$), Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. By Theorem 2.5.2, we have that the sequence is bounded. By the Bolzano-Weierstrass Theorem , the sequence has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ with limit $a \in \mathbb{R}$, that is $a_{n_k} \to a$ as $k \to \infty$. This implies that, for any $\varepsilon > 0$ there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$

$$|a_{n_k} - a| < \frac{\varepsilon}{2}.$$

Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then there exists $k_2 \in \mathbb{N}$ such that for all $n, m \geq k_2$

$$|a_n - a_m| < \frac{\varepsilon}{2}.$$

Let us now set

$$M := \max\{k_1, k_2\}.$$

Then for any $n \geq M$, the triangle inequality and the above estimates yield

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| < |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we deduce that $a_n \to a$. □

Remark 2.5.4. *From the previous theorem we note that* ***every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$****. This property called* ***(The Completeness of Real Numbers)****. Indeed, this is not true if our space is $\mathbb{Q}$. That is there exists a Cauchy sequence of rational numbers that is converges to irrational number. As we will see below.*

Consider the next theorem

Theorem 2.5.5. *Let r be a real number. Then there exists a Cauchy sequence of rational numbers which is converges to r. Namely*

$$\textit{Exists } \{a_n\}_{n=1}^{n} \textit{ Cauchy with } a_n \in \mathbb{Q} \textit{ for all } n \textit{ and } a_n \to r.$$

Proof. Let $r \in \mathbb{R}$. By the density of rational numbers, for each $n \in \mathbb{N}$ there exists a rational number r_n such that

$$r - \frac{1}{n} < r_n < r + \frac{1}{n},$$

that is

$$|r - r_n| < \frac{1}{n}.$$

Now, for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$ Archemidean property. Thus for all $n \geq k$ we have

$$|r - r_n| < \frac{1}{n} \leq \frac{1}{k} < \varepsilon.$$

Hence $r_n \to r$ and this sequence is Cauchy by Theorem 2.5.3. □

2.6 Countable and Uncountable sets

In this section we will show that the real set is uncountable set. Lets start with the following definition

Definition 2.6.1. *Let S be a non-empty set. Then S is called a* ***countable*** *if there exists an injective function $f : S \to \mathbb{N}$. And S is called* ***countably infinite*** *if it is countable and infinite. For example, the rational numbers is countable and infinite set.*

Remark 2.6.2.

• Any finite set is countable. Indeed, if $S = \{a_1, a_2, a_3, \cdots, a_n\}$. Then we can define an injective function $f : S \to \mathbb{N}$ by $f(a_n) = n$. Similarly, if the elements of S can be enumerated as a sequence, $S = \{a_1, a_2, a_3, \cdots\}$. Then S is countable. In fact, the above examples characterise countable sets.

• Suppose $\mathcal{A} = \{S_1, S_2, S_3, \cdots\}$ is a countable collection of countable sets. Then

$$\bigcup_{S \in \mathcal{A}} S = \{a : a \in S \text{ for some } S \in \mathcal{A}\}$$

is countable. (Notice $\mathcal{A}$ is a set whose elements are sets.)

Before we show that $\mathbb{R}$ is uncountable we need the following Theorem.

Theorem 2.6.3 (Nested intervals theorem)**.** *Let $I_n := [a_n, b_n]$ be a sequence of closed intervals such that*

$$I_{n+1} = [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] = I_n \qquad \text{for all } n \in \mathbb{N}.$$

Then

1. *The intersection of all these intervals is not empty ($\bigcap_n I_n \neq \emptyset$).*

2. *If the sequence $\{|I_n|\}_{n=1}^{\infty}$ converges to 0, then $\bigcap_n I_n$ contains a single point. (Where $|I_n| = b_n - a_n$, for all $n \in \mathbb{N}$).*

Theorem 2.6.4. *The set of Real numbers is uncountable.*

Proof. Suppose that S be a countable infinite set with $S \subseteq \mathbb{R}$. We will show that $S \neq \mathbb{R}$.
Let

$$S = \{a_1, a_2, \cdots, a_n, \cdots\}.$$

Suppose that I_1 is a closed interval with $|I_1| < 1$ and $a_1 \notin I_1$, and let I_2 be a closed interval with $|I_2| < \frac{1}{2}$ such that $I_2 \subseteq I_1$ and $a_2 \notin I_2$. In general, let I_n be a closed interval with $|I_n| < \frac{1}{n}$ such that $I_{n+1} \subseteq I_n$ and $a_n \notin I_n$.
Using the previous theorem we get $\bigcap_n I_n = \{x\}$. It is clear that $x \in I_n$ for all n and since $a_n \notin I_n$ for all n, then $x \notin S$. Therefore $\mathbb{R} \neq S$ and hence $\mathbb{R}$ is uncountable. □

Chapter 3

Metric Spaces

3.1 Introduction

In this chapter we shall study more general "spaces" and "functions" defined on them. We arrive at a sufficiently general and flexible concept of a "space" as follows. We replace the set of real numbers underlying $\mathbb{R}$ by an abstract set X (set of elements whose nature is left unspecified) and introduce on X a "distance function" which has only a few of the most fundamental properties of the distance function on $\mathbb{R}$.

Definition 3.1.1. Let X be a non-empty set. Suppose that $d : X \times X \to \mathbb{R}^+$ is a function. Then (X, d) is a **metric space** if the following conditions are satisfies for all $x, y, x \in X$:
(M1) d is real valued, finite and non-negative.
(M2) $d(x, y) = 0$ if and only if $x = y$.
(M3) $d(x, y) = d(y, x)$ **(Symmetry)**.
(M4) $d(x, y) \leq d(x, z) + d(z, y)$ **(Triangle inequality)**.
The function d is called a **metric on** X or **distance function on** X.

The elements of X are called **points**. For fixed $x, y \in X$ the non-negative $d(x, y)$ is called the **distance from** x **to** y.

Example 3.1.2 (Real Line $\mathbb{R}$)**.**
Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a metric space.

Let $x, y, z \in \mathbb{R}$. Then:

- It is clear that d is real valued, finite and non-negative.
- If $x = y$ then $d(x, y) = |x - y| = 0$.

- $d(x,y) = |x-y| = |y-x| = d(y,x)$.
- $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$.

Example 3.1.3 (Euclidean plane $\mathbb{R}^2$)**.**

The metric space $\mathbb{R}^2$, called the **Euclidean plane**, is obtained if we take the set of ordered pairs of real numbers, written $x = (x_1, x_2), y = (y_1, y_2)$, etc. Take d as follows

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Then $(\mathbb{R}^2, d)$ is a metric space. d is called the **Euclidean metric**.

Another metric space is obtained if we choose that same set as before but another metric d_1 defined by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Let $x, y, z \in \mathbb{R}^2$. Then

- It is clear that d_1 is real valued, finite and non-negative.
- If $x = y$ then $(x_1, x_2) = (y_1, y_2)$. Thus $d_1(x,y) = 0$.
- $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d_1(y,x)$.
- $$\begin{aligned} d_1(x,y) &= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &= d_1(x,z) + d_1(z,y) \end{aligned}$$

Example 3.1.4 (Euclidean plane $\mathbb{R}^n$)**.**

The previous example is a special case of n-dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set of all ordered n-tuples of real numbers, written

$$x = (x_1, x_2, \cdots, x_n), \qquad y = (y_1, y_2, \cdots, y_n), etc.$$

$(\mathbb{R}^n, d)$, $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are metric spaces where

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2},$$

$$\sum_{k=1}^{n} x_k + 4$$

$$\lim_{n\to\infty} a_n$$

and

$$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n| = \sum_{k=1}^{n} |x_k - y_k|$$

and

$$d_\infty(x,y) = \max_{i=1\in\mathbb{N}} |x_i - y_i|.$$

- - - -

- -

Example 3.1.6 (Discrete metric space)**.** Let X be any non-empty set. The discrete metric on X defined by

$$d(x,y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

for all $x, y \in X$. Then (X, d) is a metric space.

Let $x, y, z \in X$. Then:

- It is clear that d is real valued, finite and non-negative.
- If $x = y$ then $d(x, y) = 0$.
- $d(x, y) = 1 = d(y, x)$
- $d(x, y) = 1 < d(x, z) + d(z, y) = 2$.

Remark 3.1.7.

(1) A **subspace** $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset $Y \subseteq X$ and restrict d to $Y \times Y$, thus the metric on Y is the restriction

$$\tilde{d} = d_{Y \times Y}$$

$\tilde{d}$ is called the metric **induced** on Y by d.

(2) Let $x, y \in \mathbb{R}^n$ and $p \geq 1$ be a real number. Then the Minkowski inequality:

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{m=1}^{n} |y_m|^p \right)^{1/p} \tag{3.1}$$

(3) Let $x, y \in \mathbb{R}^n$. Then the Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \sqrt{\sum_{k=1}^{n} |x_k|^2} \cdot \sqrt{\sum_{m=1}^{n} |y_m|^2} \tag{3.2}$$

(4) Let (X, d) be a metric space. Then for $x_1, x_2, \cdots, x_n \in X$ we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

3.2 Basic Principles of Topology

There is a considerable number of auxiliary concepts which play a role in connection with metric spaces. Those which we shall need are included in this section. We first consider important types of subsets of a given metric space (X, d).

Definition 3.2.1. Let $x_0 \in X$ and r be a real number. Then we define the following sets:

i. $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ **(Ball with center x_0 and radius r).**

ii. $D(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ **(Disk with center x_0 and radius r).**

iii. $S(x_0, r) = \{x \in X : d(x, x_0) = r\}$ **(Sphere with center x_0 and radius r).**

The definition immediately implies that

$$S(x_0, r) = D(x_0, r) - B(x_0, r)$$

$B(x_0, r)$ is often called a **Neighborhood** of x_0.

Example 3.2.2.

Describe the ball, disk and sphere in each of the following metric spaces

(1) Real line $\mathbb{R}$.

$B(x_0, r) = \{x \in X : |x - x_0| < r\} = (x_0 - r, x_0 + r)$ (open interval).

$D(x_0, r) = \{x \in X : |x - x_0| \leq r\} = [x_0 - r, x_0 + r]$ (closed interval).

$S(x_0, r) = \{x \in X : |x - x_0| = r\} = \{x_0 + r, x_0 - r\}$.

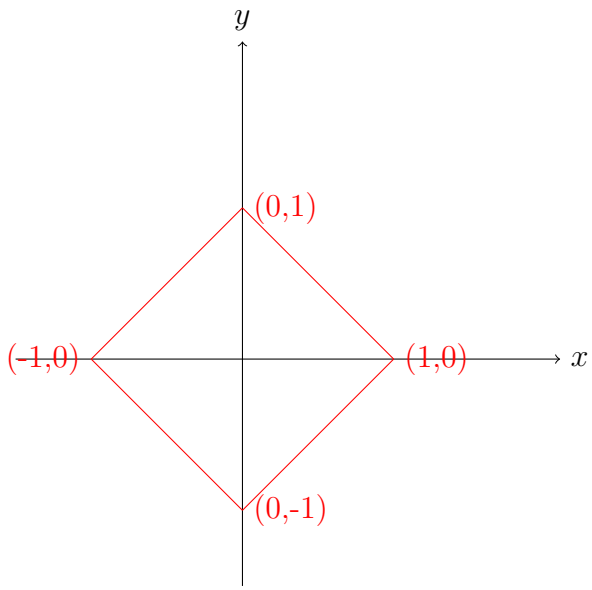
(2) Euclidean plane $\mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

$B(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$ (inside of the circle with centre y and radius r).

$D(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq r^2\}$, (inside of the circle with centre y and radius r as well as its circumference).

$S(y, r) = \left\{x \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = r\right\} = \{x \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 = r^2\}$, (circle with centre y and radius r).

(3) Euclidean plane $\mathbb{R}^2$ with $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Take $x_0 = 0$ and $r = 1$.



$B(0,1) = \{y \in \mathbb{R}^2 : |0 - y_1| + |0 - y_2| < 1\} = \{y \in \mathbb{R}^2 : |y_1| + |y_2| < 1\}$ (The ball will be all the points inside this rhombus (this shape).

$D(0,1) = \{y \in \mathbb{R}^2 : |y_1| + |y_2| \leq 1\}$. The disk will be all the points inside this rhombus (this shape) as well as its perimeter.

$S(0,1) = \{y \in \mathbb{R}^2 : |y_1| + |y_2| = 1\}$. The sphere will be all the points on the perimeter of this rhombus (shape).

Definition 3.2.3 (Open set, closed set)**.**

Let (X, d) be a metric space. A subset $S \subseteq X$ is said to be **open** if for every $x_0 \in S$ there exists $r > 0$ such that the ball $B(x_0, r) \subseteq S$. A subset $M \subseteq X$ is said to be **closed** if its complement (in X) is open, that is, $M^c = X - M$ is open.

Example 3.2.4.

(1) Let (X, d) be a metric space. If $x_0 \in X$ and $r > 0$ then $B(x_0, r)$ is an open set.

Proof:

Suppose that there exists $y_0 \in B(x_0, r)$, thus $0 \leq d(y_0, x_0) < r$. Then there exits $t > 0$ such that

$$d(y_0, x_0) = r - t.$$

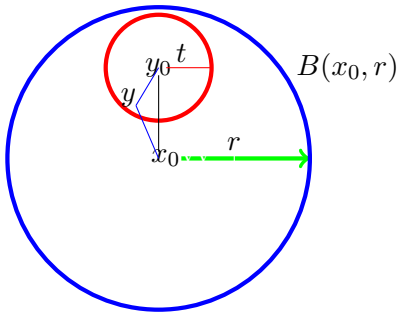
For all points $y \in b(y_0, t)$, that is, $d(y, y_0) < t$ we have

$$d(y, x_0) \leq d(y, y_0) + d(y_0, x_0) < t + r - t = r,$$

that is

$$d(y, x_0) \leq r, \text{ for all y} \in B(y_0, t).$$

Hence $y \in B(x_0, r)$.



(2) Let $X = \mathbb{R}$. For $a \in \mathbb{R}$, show that

$$(a, \infty) = \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

are open sets.

Solution:

HOMEWORK

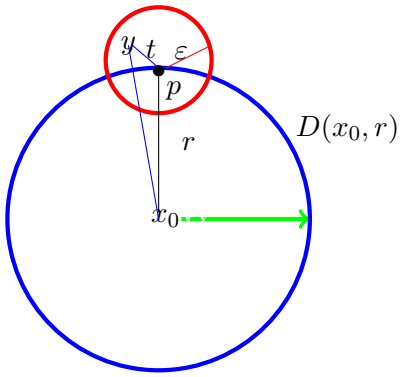
(3) Show that every disk in $\mathbb{R}^2$ (or in $\mathbb{R}^n$) is not open.

Solution:

Let $D(x_0, r)$ be a disk with centre $x_0 \in \mathbb{R}^2$ and radius $r > 0$. Suppose that $p \in D(x_0, r)$ such that $d(x, p) = r$. For any $\varepsilon > 0$, let $B(p, \varepsilon)$ be a ball around p with radius $\varepsilon > 0$.

Take $y \in B(p, \varepsilon)$ such that $d(p, y) = t < \varepsilon$. Then

$d(x_0, y) = d(x_0, p) + d(p, y) = r + t > r$. Thus $y \notin D(x_0, r)$ that is $B(p, \varepsilon) \nsubseteq D(x_0, r)$, hence $D(x_0, r)$ is not open.



(4) Show that the set of rational numbers is not open in $\mathbb{R}$.

Solution:

Homework

Lemma 3.2.5. *Let (X, d) be a metric space. Suppose that $\mathcal{T}$ be a collection of all open subsets of X. Then*

T1. *$\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.*

T2. *The union of (finite or infinite) members of $\mathcal{T}$ is a member of $\mathcal{T}$.*

T3. *The intersection of finitely many members of $\mathcal{T}$ is a member of $\mathcal{T}$.*

Proof.

T1. As the empty set contains no points, then obviously it is open. The whole space X is open, since every open ball centred at any of its points is contained in X.

T2. Let $\{G_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of open sets . Suppose that $x \in \bigcup_{\alpha\in\Lambda} G_\alpha$ then there exists at least one open set say G_α (for some $\alpha \in \Lambda$) such that $x \in G_\alpha$. Since G_α is open then there exists a ball $B(x, r) \subseteq G_\alpha$ and by the definition of a union, $B(x, r) \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$. Thus, for each $x \in \bigcup_{\alpha\in\Lambda} G_\alpha$ there exists an $r > 0$ such that $B(x, r) \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$. Consequently, $\bigcup_{\alpha\in\Lambda} G_\alpha$ is open.

T3. Let $\{G_j : 1 \leq j \leq n\}$ be a finite family of open sets in X. Suppose that $x \in \bigcap_{j=1}^{n}$ then $x \in G_j$ for all $j = 1, 2, \cdots, n$. Since each G_j is open then there exists $r_j > 0$ such that $B(x, r_j) \subseteq G_j$, $j = 1, 2, \ldots, n$. Let $r = \min\{r_1, r_2, \cdots, r_n\}$. Thus we have $r > 0$ and $B(x, r) \subseteq G_j$ for all $j = 1, 2, \cdots, n$, that is $B(x, r) \subseteq \bigcap_{i=1}^{n}$. This implies that $\bigcap_{i=1}^{n}$ is open. □

Remark 3.2.6. The intersection of an infinite number of open sets does not necessarily have to be

open. For example, let $S_n = (-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}, n = 1, 2, \cdots$. Each S_n is an open ball in $\mathbb{R}$. However,

$$\bigcap_{n=1}^{\infty} S_n = \{0\}$$

which is not open.

From the previous theorem and the definition of the open set we can prove that

Lemma 3.2.7. *Let (X, d) be a metric space. Suppose that $S \subseteq X$, then S is open if and only if it is a union of open balls.*

Consider the following examples for closed sets.

Example 3.2.8.

(1) Let $a, b \in \mathbb{R}$. Show that $[a, b]$ is a closed set in the real line $\mathbb{R}$.

Solution: Note that $[a, b]^c = (-\infty, a) \cup (b, \infty)$ and each of $(-\infty, a)$, (b, ∞) is an open set and their union is also open. Thus $[a, b]$ is a closed set.

(2) Prove that every disk in $\mathbb{R}^n$ is a closed set.

Solution: Let $x \in \mathbb{R}^n$ and $r > 0$. To prove $D(x, r)$ is closed in $\mathbb{R}^n$ we will show that its complement $G = \mathbb{R}^n \backslash D(x, r)$ is open. Suppose that $y \in G$, then $d(x, y) > r$. Put $t := d(x, y) - r > 0$, thus $r = d(x, y) - t$. Let $b \in B(y, t)$. Then

$$d(x, y) \leq d(x, b) + d(b, y)$$

$$< d(x, b) + t,$$

hence

$$d(x, y) - t < d(x, b),$$

that is $d(x, b) > r$ and so $b \notin B(x, r)$. Therefore $B(y, t) \subseteq G = \mathbb{R}^n \backslash B(x, r)$ and G is open.

(3) Show that the set of rational numbers $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Solution:

Homework

(4) Show that every finite set is closed.

Solution:

Homework

Lemma 3.2.9. *Let (X, d) be a metric space. Then:*

(1) *X and $\emptyset$ are both closed.*

(2) *The intersection of any collection (finite or infinite) of closed sets is closed.*

(3) *The union of any finite collection of closed sets is closed.*

Proof. Homework

□

Question: Is the union of infinite collection of closed sets is closed? Why? explain your answer by an example.

We shall now introduce concepts, that are related with open and closed sets.

Definition 3.2.10 (Limit point)**.**

Let (X, d) be a metric space. Suppose that $S \subseteq X$, then $x_0 \in X$ (x_0 may or may not be a point of S) is said to be a **limit point** (or an **accumulation point**) for S if every ball with centre x_0 contains at least one point $p \in S$ distinct from x_0. That is for every $B(x_0, r)$

$$(B(x_0, r) - \{x_0\}) \bigcap S \neq \emptyset.$$

We will use the symbol S' to refer to the set of all limit points of S. The **closure** of S consisting of all points of S and the limit points of S, that is

$$\bar{S} = S \cup S'$$

The closure of S is the smallest closed set containing S.

Example 3.2.11. Let $S = (a, b) \subseteq \mathbb{R}$. Show that $S' = [a, b]$.

Let $x \in (a, b)$. Then x is a limit point of $S = (a, b)$. Let $r > 0$. Then the interval $(x - r, x + r)$ will contain points of (a, b) other than x. In fact $(x - r, x + r)$ contains infinitely many points of (a, b) distinct from x. Moreover, a and b are limit points of (a, b). Let $\delta > 0$. Then the interval $(a - \delta, a + \delta)$ contains infinitely many points of (a, b) distinct from a. That is

$$(a - \delta, a + \delta) \setminus \{a\} \bigcap (a, b) \neq \emptyset,$$

thus a is a limit point for (a, b). The same is true for b. Hence $S' = [a, b]$ and $\bar{S} = S \cup S' = (a, b) \cup [a, b] = [a, b]$

Example 3.2.12. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\} \subseteq \mathbb{R}$. Show that $S' = \{0\}$.

For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ (by Archimedean property) such that $\frac{1}{n} < \varepsilon$. Thus the interval $(-\varepsilon, \varepsilon)$ contains $\frac{1}{n} \in S$ and $\frac{1}{n} \neq 0$. In other words

$$((-\varepsilon, \varepsilon) \setminus \{0\}) \bigcap S \neq \emptyset.$$

This shows that $0 \in S'$.

Now, we have to show that if $x \neq 0$ then $x \notin S'$. Note that if $x < 0$ or $x \geq 1$ then there exists $\varepsilon > 0$ such that

$$((x - \varepsilon, x + \varepsilon) - \{x\}) \cap S = \emptyset.$$

Hence, $x \notin S'$. If $0 < x < 1$ we have two cases, **either** $x \in S$, that is $x = \frac{1}{n}$ for some $n \in \mathbb{N}$. Hence

$$\left((\frac{1}{n+1}, \frac{1}{n-1}) - \{x\} \right) \cap S = \emptyset$$

Thus $x \notin S'$. **Or** $x \notin S$, this implies that $\frac{1}{x} \notin \mathbb{N}$. Hence there exists $n \in \mathbb{N}$ such that $n < \frac{1}{x} < n + 1$,

that is, $\frac{1}{n+1} < x < \frac{1}{n}$. Hence

$$\left((\frac{1}{n+1}, \frac{1}{n}) - \{x\}\right) \cap S = \emptyset.$$

Therefore $x \notin S'$ and 0 is only the limit point for S.

Example 3.2.13. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$. Then $\mathbb{Q}' = \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

Suppose that $x_0 \in \mathbb{R}$ then for each open interval around x_0 contains many rational numbers (by the density of the rational numbers). Thus $\mathbb{Q}' = \mathbb{R}$. Hence $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

Example 3.2.14. Let S be a finite set in $\mathbb{R}^n$. Then $S' = \emptyset$.

Solution: Homework

Example 3.2.15. Let $\mathbb{Z}$ be the set of all integer numbers in $\mathbb{R}$. Then $\mathbb{Z}' = \emptyset$.

Solution:

The same as the previous example.

Theorem 3.2.16. *Let X be a metric space (or a Topological space). Then $S \subseteq X$ is closed if and only $S' \subseteq S$. That is $S \subseteq X$ is closed if and only if it contains all its limit points. (or S is closed if and only if $S = \bar{S}$).*

Proof. ($\Rightarrow$) Suppose that S is closed, then by the definition its complement $X \backslash S$ is open. Take $x \notin S$, that is $x \in X \backslash S$. Since is open then there is $r > 0$ such that the ball $B(x, r) \subseteq X \backslash S$. Hence $B(x, r) \cap S = \emptyset$, this implies that x is not a limit point for S. Thus $S' \subseteq S$.

($\Leftarrow$) For the converse, suppose that $S' \subseteq S$. To prove S is closed we will show that $X \backslash S$ is open. Let $x \in X \backslash S$. Since x is not a limit point for S then there exists $r > 0$ such that $B(x, r) \cap S = \emptyset$. That is $B(x, r) \subseteq X \backslash S$. Hence $X \backslash S$ is open and S is closed. □

Definition 3.2.17 (The distance from a point to a set)**.**
Let (X, d) be a metric space. Suppose that S be a non-empty subset of X and $p \in X$. Put

$$A = \{d(p, x) : x \in S\}.$$

Since d is non-negative, then A is bounded below by 0, thus A has greatest lower bound. Set

$$d(p, S) = \inf A = \{d(p, x) : x \in S\},$$

then $d(p, S)$ is called the **distance** of p from the set S.

Remark 3.2.18.

1. $d(p, S) \geq 0$.
2. If $p \in S$ then $d(p, S) = 0$.
3. If $S = \{x\}$ then $d(p, S) = d(p, x)$.

Example 3.2.19. Let $S = (a, b)$ in $\mathbb{R}$. Show that $d(a, S) = 0$ and $d(b, S) = 0$.

Solution:

Let $x \in S$. Then $d(a, x) = |x - a|$ for all $x \in S$. That is

$$A = \{d(a, x) : x \in S\} = \{|x - a| : x \in S\}.$$

But $\inf A = 0$, hence $d(a, S) = 0$.

By the same way we can show that $d(b, S) = 0$.

From the previous example we note that if $d(p, S) = 0$ this does note means that $p \in S$.

Lemma 3.2.20. *Let (X, d) be a metric space. Suppose that $S \subseteq X$ be a non-empty set and $p \in X$. Then $d(p, S) = 0$ if and only if $p \in S$ or p is a limit point for S.*

3.3 Convergence in Metric Spaces

Definition 3.3.1. Let (X, d) be a metric space and $\{a_n\}_{n=1}^{\infty}$ be a sequence in X. We say that $\{a_n\}_{n=1}^{\infty}$ **converges** to $a_0 \in X$ as $n \to \infty$, if for any $\varepsilon > 0$ there exists an integer $k \in \mathbb{N}$ such that

$$d(a_n, a_0) < \varepsilon \qquad \text{for all} \quad n \geq k.$$

a_0 is called the convergence point for the sequence $\{a_n\}_{n=1}^{\infty}$ and we write

$$a_n \to a_0 \qquad \text{or} \qquad \lim_{n\to\infty} a_n = a_0.$$

In other words, $\{a_n\}_{n=1}^{\infty}$ converges to a_0 if for every ball with centre a_0 contains most of the terms of $\{a_n\}_{n=1}^{\infty}$ with exclusion of finite number of terms.

Lemma 3.3.2. *Let (X, d) be a metric space. Then:*

1. *The limit of a convergent sequence in X is unique.*

2. *A convergent sequence in X is bounded.*

3. *If $a_n \to a_0$ and $b_n \to b_0$ in X, then $d(a_n, b_n) \to d(a_0, b_0)$.*

Proof. (1)
Suppose that a_0 and b_0 are two convergence points for $\{a_n\}_{n=1}^{\infty}$ such that $a_0 \neq b_0$. Let $r = d(a_0, b_0) > 0$. Since $a_n \to a_0$ then the ball $B(a_0, \frac{r}{2})$ contains most of the terms of $\{a_n\}_{n=1}^{\infty}$. Also, since $a_n \to b_0$ then the ball $B(b_0, \frac{r}{2})$ contains most of the terms of $\{a_n\}_{n=1}^{\infty}$. But $B(a_0, \frac{r}{2}) \cap B(b_0, \frac{r}{2}) = \emptyset$ which is a contradiction. Thus $a_0 = b_0$. □

Proof. (2) and (3) Homework

□

Theorem 3.3.3. *Let S be a non-empty subset of a metric space (X, d) and $\bar{S}$ be the closure of S. Then:*

1. *$x_0 \in \bar{S}$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \to x_0$.*

This means that if $x_0 \in S$ or x_0 is a limit point for S ($x_0 \in S'$), then there exists a sequence $x_n \in S$ such that $x_n \to x_0$ and conversely, if x_n is a sequence in S with $x_n \to x_0$ then $x_0 \in S$ or $x_0 \in S'$.

2. *S is closed if and only the case $x_n \in S$ with $x_n \to x_0$ implies that $x_0 \in S$.*

Proof. Homework □

Cauchy Sequences (Fundamental Sequences)

Definition 3.3.4. Let (X, d) be a metric space. Then the sequence $\{a_n\}_{n=1}^{\infty}$ is called a **Cauchy sequence** if for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \quad \text{for all} \quad n, m > k.$$

Theorem 3.3.5. *Let (X, d) be a metric space. Then every convergent sequence in X is a Cauchy sequence.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence in X such that $a_n \to a_0 \in X$. That is for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_0) < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq k.$$

Thus for all $n, m \geq k$

$$d(a_n, a_m) \leq d(a_n, a_0) + d(a_m, a_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequences. □

The converse of this theorem is not true. That is if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d) then $\{a_n\}_{n=1}^{\infty}$ may not converge in X. Consider the following examples:

Example 3.3.6.

(1) Let $X = \mathbb{R}\setminus\{0\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a metric space. Suppose that $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$, that is $a_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Then a_n is a Cauchy sequence (Why? show that) and $a_n \to 0$ but $0 \notin X$. Hence a_n does not converge in X.

(2) Let $X = \mathbb{Q}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a metric space. We have noted that for every real number r there exists a Cauchy sequence of rational numbers r_n such that $r_n \to r$. Suppose that $r = \sqrt{2}$, thus $r_n \to \sqrt{2}$. But $\sqrt{2} \notin X$, thus r_n does not converge in X.

Complete Metric Spaces

Definition 3.3.7. Let (X, d) be a metric space. Then (X, d) is called a **complete metric space** if every Cauchy sequence in X is convergent in X.

From that previous example we note that $(\mathbb{R}\setminus\{0\}, d)$ and $(\mathbb{Q}, d)$ are not complete metric spaces.

Example 3.3.8. Prove that the Euclidean space $(\mathbb{R}^2, d)$ is complete metric space.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^2$ and let $\varepsilon > 0$. Suppose that $a_n = (x_n, y_n)$ for all $n \in \mathbb{N}$, where $x_n, y_n \in \mathbb{R}$. Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence then there exists $k \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \quad \text{for all } n, m \geq k$$

that is

$$\sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \varepsilon$$

$$(x_n - x_m)^2 + (y_n - y_m)^2 < \varepsilon^2$$

This implies that

$$|x_n - x_m| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad |y_n - y_m| < \frac{\varepsilon}{\sqrt{2}} \quad \text{for all } n, m \geq k.$$

That is $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space then there exists $x_0, y_0 \in \mathbb{R}$ such that

$$x_n \to x_0 \quad \text{and} \quad y_n \to y_0,$$

That is there exists $k \in \mathbb{N}$ such that

$$|x_n - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y_n - y_0| < \frac{\varepsilon}{2} \quad \text{for all } \ n \geq k.$$

Set $a_0 = (x_0, y_0)$, then

$$\begin{aligned}[d(a_n, a_0)]^2 &= (x_n - x_0)^2 + (y_n - y_0)^2 \\ &< \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2} \quad \text{for all } \ n \geq k,\end{aligned}$$

hence $d(a_n, a_0) < \varepsilon$ for all $n > k$, thus $a_n \to a_0$ and $\mathbb{R}^2$ is a complete metric space. □

Lemma 3.3.9. *Let (X, d) be a complete metric space. Suppose that S is a non-empty closed subset of X, then (S, d) is complete metric space.*

Definition 3.3.10 (Bounded sets)**.**
Let (X, d) be a metric space. A non-empty S subset of X is said to be **bounded** if for every $x_0 \in S$, there exists $r > 0$ such that

$$d(x, x_0) < r \quad \text{for all } x \in S.$$

In other words S is bounded if there exists $r > 0$ such that $S \subseteq B(x, r)$ for some $x \in S$. *Geometrically* this means that S is bounded if and only if there exists a ball with centre x_0 contains S.

Definition 3.3.11 (Diameter of a set)**.**

Let (X, d) be a metric space and S be a non-empty subset of X. Set

$$\hat{S} = \{d(x, y) : x, y \in S\}$$

$\hat{S}$ is a subset of $\mathbb{R}$ and it is bounded below by 0. If S is bounded then $\hat{S}$ is bounded above and

$$D(S) = \text{diam}(S) = \sup(\hat{S})$$

is called the **diameter** of S

Lemma 3.3.12. *Let (X, d) be a metric space. A non-empty subset S of X is bounded if and only if $\hat{S}$ is bounded above.*

Proof. Homework

□

Example 3.3.13.

(1) Let X be the euclidean metric space $\mathbb{R}$. Then $\mathbb{R}$ and $\mathbb{Q}$ are unbounded set.

For $a, b \in \mathbb{R}$, the sets $[a, b], (a, b)$ are bounded and the diameter for each one is $b - a$. Why?

(2) Let X be the euclidean metric space $\mathbb{R}^n$. For $r > 0$ and $x_0 \in \mathbb{R}^n$ let $S = B(x_0, r)$. Then S is bounded and its diameter is $2r$. Why?

Note that for any $x, y \in B(x_0, r)$ we have $d(x, y) \leq d(x, x_0) + d(x_0, y) < r + r = 2r$. Thus

$$\hat{S} = \{d(x, y) : x, y \in B(x_0, r)\} < 2r$$

and $\sup(\hat{S}) = 2r$. Hence $\text{diam}(B(x_0, r))$ is $2r$.

Let $D(x_0, r)$ be a disk in $\mathbb{R}^n$. Show that $D(x_0, r)$ is bounded set and find its diameter.

(3) Let X be any non-empty set. The discrete metric on X defined by

$$d(x, y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

Let $x_0 \in X$ and

$$S = \left\{ x \in X : d(x, x_0) < \frac{1}{2} \right\}.$$

Note that S has only one point x_0, thus S is bounded. Moreover,

$$\hat{S} = \{d(x, y) : x, y \in S\} = 0$$

hence $\sup(S) = 0$ and $\text{diam}(S) = 0$.

Theorem 3.3.14 (Cantor's Nested Sets Theorem)**.**

Let (X, d) be a metric space. Suppose that $\{E_n\}_{n=1}^{\infty}$ is a sequence of bounded subset of X such that

(1) *$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots \supseteq E_n \supseteq \cdots$*

(2) *E_n is non-empty and closed for all $n \in \mathbb{N}$*

(3) *$D(E_n) = diam(E_n) \to 0$ as $n \to \infty$.*

If (X, d) is a complete metric space then

$$\bigcap_n E_n \neq \phi,$$

that is, it is not empty and contains only one point.

Contraction Mapping

Definition 3.3.15. Let (X, d) be a metric space. A **fixed point** of a mapping $T : X \to X$ of a set X into itself is a point $x_0 \in X$ which is mapped onto itself (is "kept fixed" by T), that is,

$$Tx_0 = x_0$$

the image Tx_0 coincides with x_0.

Example 3.3.16. Let X be the euclidean metric space $\mathbb{R}$. suppose that $T : \mathbb{R} \to \mathbb{R}$ be a function defined by $T(x) = x^2$. Then T has two fixed point 0 and 1.

Definition 3.3.17. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a **contraction mapping** on X if there is a real number $0 \leq \alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Geometrically this means that any points x and y have images that are closer together than those points x and y.

Example 3.3.18. Let X be the euclidean metric space $\mathbb{R}$. Then $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = \frac{x}{2}$, then T is a contraction mapping.

Solution: Let $x, y \in \mathbb{R}$, then

$$\begin{aligned} d(T(x), T(y)) &= d\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= \left|\frac{x}{2} - \frac{y}{2}\right| \\ &= \frac{1}{2}|x - y| \\ &= \frac{1}{2}d(x, y), \end{aligned}$$

thus $d(Tx, Ty) = \alpha d(x, y)$, where $\alpha = 0.5$.

Theorem 3.3.19 (Fixed point theorem)**.**
Let (X, d) be a complete metric space. If $T : X \to X$ is a contraction mapping then there exists only one point $x_0 \in X$ such that:

$$Tx_0 = x_0,$$

that is the equation $Tx_0 = x_0$ has only one solution.

3.4 Compact Metric Spaces

Definition 3.4.1. Let (X, d) be a metric space. An **open cover** of S in X is a collection of open sets $\{G_\alpha\}$ whose union "covers" (contains) S. That is

$$S \subseteq \bigcup_\alpha G_\alpha.$$

A **subcover** of $\{G_\alpha\}$ is a subcollection $\{G_{\alpha_i}\}$ that still covers S.

Example 3.4.2. Let $S = [\frac{1}{2}, 1)$ in $\mathbb{R}$. Then S has a cover $\{G_n\}_{n=3}^{\infty}$ where $G_n = (\frac{1}{n}, 1 - \frac{1}{n})$.

$G_3 = (\frac{1}{3}, 1 - \frac{1}{3}) = (\frac{1}{3}, \frac{2}{3})$

$G_4 = (\frac{1}{4}, 1 - \frac{1}{4}) = (\frac{1}{4}, \frac{3}{4})$

$G_5 = (\frac{1}{5}, 1 - \frac{1}{5}) = (\frac{1}{5}, \frac{4}{5})$

:

:

$G_{10} = (\frac{1}{10}, 1 - \frac{1}{10}) = (\frac{1}{10}, \frac{9}{10})$

:

:

Also, $\{(0, 2)\}$ is an open cover of S contains one element.

Also, $\{W_x\}_{x \in [\frac{1}{2}, 1)}$, where $W_x = B(x, 0.1)$ is an open cover for S.

In each cover, do we need all these sets to still cover S?

Note that:

$\{G_n\}_{n=3}^{\infty}$ has finite subcover $\{G_n\}_{n=122}^{\infty}$ and is still cover the set S.

$\{(0, 2)\}$ has finite subcover $\{(0, 2)\}$ and this still covers S.

$\{W_x\}_{x \in [\frac{1}{2}, 1)}$ has finite subcover $\left\{W_{\frac{5}{10}}, W_{\frac{6}{10}}, W_{\frac{7}{10}}, W_{\frac{8}{10}}, W_{\frac{9}{10}}\right\}$ and this still covers S.

Example 3.4.3. Let $S = [0, 1]$ in $\mathbb{R}$. Then S has a cover $\{G_n\}_{n=3}^{\infty} \cup \{W_0, W_1\}$. This cover has a finite subcover $\{W_0, W_1, G_{11}\}$, and this still covers S.

Definition 3.4.4. Let (X, d) be a metric space. A set S is **compact** in X, if for every open cover of S contains a finite subcover.

*So, S is **not compact** means, there exists some open cover that has no finite subcover.*

If $S = X$ then we say that X is a **compact space**.

Example 3.4.5. $S = (0, 1)$ in $\mathbb{R}$ is not compact. We can not find a finite subcover for $\{G_n\}_{n=3}^{\infty}$, that still cover S.

Example 3.4.6. $\mathbb{Z}$ in $\mathbb{R}$ is not compact.

Homework

Example 3.4.7. Show that $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ is not compact in $\mathbb{R}$.

Homework

Example 3.4.8. $[0, 1]$ in $\mathbb{R}$, may be compact? we need to check every open cover? (or prove theorem).

Lemma 3.4.9. *Finite sets are compact.*

Proof. Let (X, d) be a metric space and $S \subseteq X$ be a finite set. Consider an open cover $\{G_\alpha\}$ covering $S = \{x_1, x_2, \cdots x_n\}$. For all x_i choose one G_{α_i} that contains x_i. Then

$$\{G_{\alpha_i}\}_{i=1}^n$$

covers the set S and it is finite subcover from $\{G_\alpha\}$. □

Theorem 3.4.10. *Compact sets are bounded.*

Proof. Let (X, d) be a metric space and S be a compact set in X.

Consider B_x be a ball with radius 1, that is $B_x = B(x, 1)$, then $\{B_x\}_{x \in S}$ is an open cover for S.

By compactness of S there exists a finite subcover $\{B_{x_i}\}_{i=1}^n$. Set

$$R = \max_{1 \leq i,j \leq n} \{d(x_i, x_j)\}$$

this maximum exists because the set $\{x_1, x_2, \cdots, x_n\}$ is finite. Then $B(x_1, R + 2)$ contains S, hence S is bounded. □

Theorem 3.4.11. *Compact sets are closed.*

Proof. No Proof

□

Example 3.4.12. Note that $S = (0, 1)$ in $\mathbb{R}$ is not compact. From the previous theorem, if S is compact it should be closed. Thus it is not compact although it is bounded.

Example 3.4.13. Note that $\mathbb{R}$ in $\mathbb{R}$ is not compact although it is closed. Because it is not bounded. Also, $S = [1, \infty)$ is closed but not compact.

So closed sets are not compact in general. Consider the next theorem.

Theorem 3.4.14. *A closed set S in a compact metric space (X, d) is compact*

Proof. Homework

□

Corollary 3.4.15. *If F is a closed in a compact set S in a metric space (X, d), then $F \cap S$ is compact.*

Proof. Homework

□

Theorem 3.4.16. *$S = [a, b]$ is compact in $\mathbb{R}$*

Proof. Suppose that S is not compact. Then there exists an open cover $\{G_\alpha\}$ that has no finite subcover. (We will try to find a contradiction).
Choose $c_1 \in S$ to split the interval in half. Then $\{G_\alpha\}$ covers both $[a, c_1]$ and $[c_1, b]$ and at least one of these intervals has no finite subcover say $I_1 = [a, c_1]$. Then subdivided again by choosing $c_2 \in [a, c_1]$, we get $[a, c_2]$, $[c_2, c_1]$ and at least one of them has no finite subcover, say $I_2 = [a, c_2]$. Continue, we obtain sequence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

of nested close intervals, and non- of them has finite subcover of $\{G_\alpha\}$, also $|I_n| \to 0$ as $n \to \infty$.

By the nested intervals theorem there exists x in I_n for all n but each I_n is closed, thus x there exists G_{α_x} that contains x. But G_{α_x} is open, that is there exists $r > 0$ such that $B(x, r) \subseteq G_{\alpha_x}$. Since $|I_n| \to 0$ as $n \to \infty$, then some $I_n \subseteq B(x, r)$ meaning G_{α_x} covers I_n which a contradiction. Hence $[a, b]$ is compact. □

The previous theorem is also true in $\mathbb{R}^n$.

Theorem 3.4.17 (Hiene-Borel Theorem)**.** *In $\mathbb{R}$ (or $\mathbb{R}^n$) S is compact if and only if S is both closed and bounded.*

Proof.
($\Rightarrow$) Suppose that S is compact in $\mathbb{R}$ (or $\mathbb{R}^n$), then S is bounded and closed by Theorem 3.4.10 and Theorem 3.4.11.
($\Leftarrow$) (NOT TRUE IN ARBITRARY METRIC SPACE)
Let S be bounded and closed in $\mathbb{R}$. Then there exists $r > 0$ such that $S \subseteq (-r, r) \subseteq [-r, r]$. Since $[-r, r]$ is closed then $[-r, r]$ is compact by previous theorem. But S is closed, hence S is compact by Theorem 3.4.14. □

Example 3.4.18. *Discrete metric space on infinite set A. A is closed and bounded but not compact.*

Theorem 3.4.19. *Let (X, d) be a metric space. S is compact in X if and only if every infinite subset of S has a limit point in S.*

Corollary 3.4.20 (Bolzano-Weierstrass Theorem)**.**

Every bounded infinite subset of $\mathbb{R}^n$ has a limit point

Proof. Let E be a bounded infinite set in $\mathbb{R}^n$. Then there exists $x \in \mathbb{R}^n$ and $r > 0$ such that $E \subseteq B(x, r) \subseteq D(x, r)$. But $D(x, r)$ is closed and bounded in $\mathbb{R}^n$, hence by Hiene-Borel Theorem $D(x, r)$ is compact. Thus by previous theorem E has a limit point in $D(x, r)$. □

Definition 3.4.21 (Dense Sets)**.**

Let (X, d) be a metric space and let $S \subseteq X$. Then S is **dense** in X if and only if $\bar{S} = X$. That is for every $x \in X$ and for every $r > 0$ we have that $B(x, r) \cap S \neq \emptyset$, i.e., every open ball in (X, d) contains a point of S.

For example $\mathbb{Q}$ is dense in $\mathbb{R}$.

SOLUTION:

Let A, B be any two compact sets in any metric space (X, d). To show that $A \cup B$ is compact, let $\{G_\alpha\}$ be an open cover of $A \cup B$. Then $\{G_\alpha\}$ is an open cover of A and B. Since A and B are compact, then $\{G_\alpha\}$ has finite subcove of A and $\{G_\alpha\}$ has finite subcover of B. The union of these subcovers, which is finite, is a subcover for $A \cup B$. Thus $A \cup B$ is compact.

Each A and B is closed and bounded, so $A \cap B$ is closed. (Lemma 3.2.9). Moreover, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so we have $A \cap B$ is closed set contained in compact set, therefore $A \cap B$ is compact (Theorem 3.4.14).

(Q7) Show that the set $B = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ is not compact.

SOLUTION:

Chapter 4

Continuity

When we studied real-valued functions of a real variable in calculus, the techniques and theory built on properties of continuity, differentiability, and integrability. All of these concepts are defined using the precise idea of a limit. In this chapter, we want to look at continuous functions on metric spaces.

4.1 Definitions and Concepts

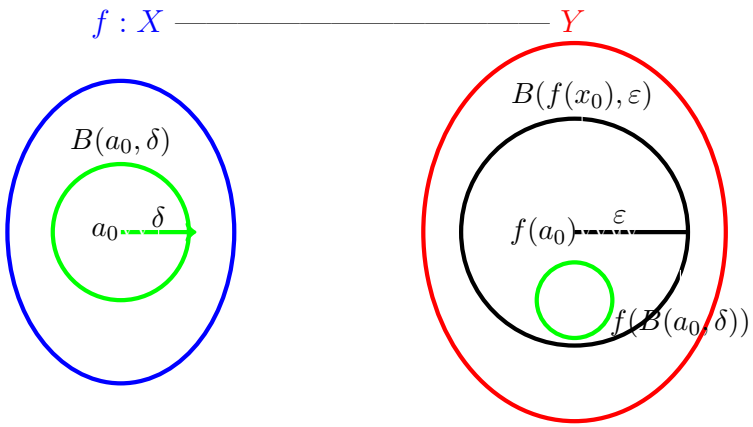
Recall that, for a function f to be continuous at a_0 we require Three things: f must be defined at a_0; $\lim_{x\to a_0} f(x)$ must be exist; and thirdly that the limit equal $f(a_0)$.

Definition 4.1.1. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is **continuous** in $a_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ (depends on ε, a_0), such that

$$\text{if} \quad d_1(x, a_0) < \delta \qquad \text{then} \qquad d_2(f(x), f(a_0)) < \varepsilon, \qquad \text{for all } x \in X.$$

In other words, $f : X \to Y$ is **continuous** in $a_0 \in X$ if for each ball $B(f(a_0), \varepsilon) \in Y$ there exists $B(a_0, \delta) \in X$ such that

$$f(B(a_0, \delta)) \subseteq B(f(a_0), \varepsilon)$$



We say that f is **continuous** if it is continuous at every point of X.

If f is not continuous at $a_0 \in X$, we say that f is **discontinuous** at a_0.

Example 4.1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x$, for all $x \in \mathbb{R}$. Then f is continuous.

Let $\varepsilon > 0$. Suppose that $a \in \mathbb{R}$, then

$$d(f(x), f(a)) = |x - a|$$

also

$$d(x, a) = |x - a|$$

thus there exists $\delta > 0$ with $\delta = \varepsilon$, such that

$$\text{if } d(x, a) = |x - a| < \delta \implies d(f(x), f(a)) = |x - a| < \varepsilon.$$

The next theorem characterizes continuous functions.

Theorem 4.1.3. *Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

That is, f is continuous if for any open set V in Y,

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open set in X.

Example 4.1.4. Show that the absolute function f on $\mathbb{R}$, i.e. $f(x) = |x|$ for all $x \in \mathbb{R}$ is continuous.

Solution: Suppose that $V = (a, b)$ is an open interval in $\mathbb{R}$, then

$$f^{-1}(V) := \begin{cases} \emptyset, & \text{if } a < b \leq 0, \\ (-b, b), & \text{if } a < 0 < b, \\ (-b, -a) \cup (a, b) & \text{if } 0 \leq a < b \end{cases}$$

we note that in each case $f^{-1}(V)$ is open in $\mathbb{R}$, thus f is continuous.

Example 4.1.5. Let X be any non-empty set. The discrete metric on X defined by

$$d(x, y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

Show that every function $f : X \to X$, is continuous.

Solution: For any V open subset of Y, its inverse image $f^{-1}(V)$ is open subset of X, since every subset of a discrete metric space is open.

Theorem 4.1.6. *Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous if and only if the inverse image of any closed subset of Y is closed subset of X.*

That is, f is continuous if for any closed set S in Y,

$$f^{-1}(S) = \{x \in X : f(x) \in S\}$$

is closed set in X.

Proof is Homework

Example 4.1.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, defined by

$$f(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases}$$

Show that f is discontinuous at $x = 0$.

Solution: Suppose that $V = (\frac{-1}{2}, \frac{1}{2})$, then $f^{-1}(V) = \{0\}$. It is clear that f is not continuous at $x = 0$.

Prove that f is continuous for all $x > 0$.

Not that $f(x) = 1$ for all $x > 0$, thus if V is any open set in $\mathbb{R}$ that contains 1 then

$$f^{-1}(V) := \begin{cases} (0, \infty), & \text{if } -1, 0 \notin V, \\ [0, \infty), & \text{if } 0 \in V, -1 \notin V, \\ \mathbb{R} \setminus \{0\}, & \text{if } -1 \in V, 0 \notin V, \\ \mathbb{R} & \text{if } -1, 0 \in V. \end{cases}$$

In each case there exists open set U contains $x \in (0, \infty)$ such that

$$f(U) \subseteq V.$$

Example 4.1.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 2x - 1$. Prove that f is continuous for all $x \in \mathbb{R}$.

Solution: Suppose that $a_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = |2x - 2a_0| = 2|x - a_0|$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta < \frac{\varepsilon}{2}$, then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon, \qquad \text{for all } x \in \mathbb{R}.$$

Thus f is continuous function on $\mathbb{R}$.

Example 4.1.9. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$. Prove that f is continuous for all $x \in (0, \infty)$.

Solution: Suppose that $a_0 \in (0, \infty)$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = |\sqrt{x} - \sqrt{a_0}| = \left| \frac{x - a_0}{\sqrt{x} + \sqrt{a_0}} \right| \leq \frac{|x - a_0|}{\sqrt{a_0}}$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta < \sqrt{a_0}\ \varepsilon$, then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon, \qquad \text{for all } x \in \mathbb{R}.$$

Thus f is continuous function on $(0, \infty)$.

Example 4.1.10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2$. Prove that f is continuous for all $x \in \mathbb{R}$.

Solution: Suppose that $a_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = \left|x^2 - a_0^2\right| = |(x - a_0)(x + a_0)|$$

$$= |x - a_0||x - a_0 + 2a_0| \leq |x - a_0|\left(|x - a_0| + 2|a_0|\right)$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta \leq 1$ then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < |x - a_0|\left(1 + 2|a_0|\right),$$

therefore if

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|a_0|}\right\}$$

then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon.$$

Thus f is continuous function on $\mathbb{R}$.

The next theorem explains the relation between the continuity and convergent.

Theorem 4.1.11. *Let (X, d) be a metric space and $f : X \to X$ be a function. Suppose that $a_0 \in X$, then f is continuous at a_0 if and only if* ***for every convergent sequence*** $\{a_n\}_{n=1}^{\infty}$ ***to*** a_0 ***in*** X**,** $(a_n \to a_0)$**, the sequence** $\{f(a_n)\}_{n=1}^{\infty}$ ***converges to*** $f(a_0)$**,** $(f(a_n) \to f(a_0))$.

Example 4.1.12. Let $f : [a, b] \to \mathbb{R}$ be a function, defined by

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 2 & \text{if } x \in \mathbb{Q}' \end{cases}$$

Show that f is discontinuous at all points $x \in [a, b]$.

Solution:

Let a_0 be a rational number, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of irrational numbers such that

$$a_n \to a_0$$

also $f(a_n) = 2$ for all $n \in \mathbb{N}$ but $f(a_0) = 1$. Thus

$$f(a_n) \nrightarrow f(a_0)$$

hence using the previous theorem, f is discontinuous at a_0.

If a_0 is an irrational number, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers such that

$$a_n \to a_0$$

also $f(a_n) = 1$ for all $n \in \mathbb{N}$ but $f(a_0) = 2$. Thus

$$f(a_n) \nrightarrow f(a_0)$$

hence using the previous theorem, f is discontinuous at a_0.

Homework:

Use the definition to show that the function f in the previous example is discontinuous.

Theorem 4.1.13. *Let $(X, d_1), (Y, d_2)$ and (Z, d_3) be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. If f is continuous at $a_0 \in X$ and g is continuous at $f(a_0) \in Y$ then the function $g \circ f$ is continuous at a_0, where $g \circ f(x) = g(f(x))$*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in X such that $a_n \to a_0$. Since f is continuous at a_0 then

$$f(a_n) \to f(a_0) \qquad \text{(by theorem 1.1.11)}$$

also since g is continuous at $f(a_0)$ then

$$g(f(a_n)) \to g(f(a_0)) \qquad \text{(by theorem 1.1.11)}$$

that is

$$g \circ f(a_n) \to g \circ f(a_0)$$

thus $g \circ f$ is continuous at a_0 by theorem 1.1.11. □

4.2 Real Valued Functions-The space $C(X)$

In this section we will suppose that X is a metric space and we study the continuous functions $f : X \to \mathbb{R}$, (The continuous real valued functions). We will use $C(X)$ to represent this set. It is clear that $C(X)$ is non-empty set, because it it contains the constant function. The next theorem shows that $C(X)$ is a metric space.

Theorem 4.2.1. *Let f, g be continuous functions from X to $\mathbb{R}$. Then*

1. *$f + g$ is a continuous function where $(f + g)(x) = f(x) + g(x)$.*
2. *fg is a continuous function where $(fg)(x) = f(x)g(x)$.*
3. *For all $a \in \mathbb{R}$, the function af is continuous where $(af)(x) = af(x)$.*
4. *If $g(a_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at a_0 where*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

5. *The function $|f|$ is continuous where $|f|(x) = |f(x)|$.*

Proof. 1.

Let $a_0 \in X$ and $\{a_n\}_{n=1}^{\infty}$ be a sequence in X that converges to a_0, $(a_n \to a_0)$. Since f is continuous, then

$$f(a_n) \to f(a_0)$$

and also

$$g(a_n) \to g(a_0)$$

using theorem 2.3.4 we get

$$f(a_n) + g(a_n) \to f(a_0) + g(a_0)$$

that is

$$(f+g)(a_n) \to (f+g)(a_0)$$

thus $f+g$ is continuous at a_0. □

Proof. 2,3,4,5 Homework □

4.2.1 Real Continuous Functions Defined on Compact Spaces

Definition 4.2.2. Let $f : X \to \mathbb{R}$ be a function. Then f is called bounded if there exists $M > 0$ such that

$$f(x) \leq M, \qquad \text{for all } x \in X.$$

In other words, a function f is **bounded** if the range of f is bounded set in $\mathbb{R}$, where

$$\text{range of } f = \{f(x) : x \in X\}.$$

Lemma 4.2.3. *Let $(X,d),(Y,d_1)$ be metric spaces and $f : X \to Y$ be a continuous function. If X is a compact space then $f(X)$ is a compact set, where*

$$f(X) = \{f(x) : x \in X\}.$$

Theorem 4.2.4. *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact then f is bounded.*

Remark 4.2.5.

1. If $f : X \to \mathbb{R}$ is a continuous function and X is not compact, then it is not necessary that f be bounded. For example

$$f(x) = \frac{1}{x}, \qquad x \in (0,\infty)$$

is continuous but not bounded. In particular for any $M > 0$ in $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that

$$f(n) = \frac{1}{n} > M. \qquad \text{(Archimedean Property).}$$

2. It is possible that a function f be bounded and continuous however X is not compact. For example:

$$f(x) = 2x \qquad \text{for all } x \in (0,1)$$

f is bonded and continuous, but $(0,1)$ is not compact.

Definition 4.2.6. Let $f : X \to \mathbb{R}$ be a bounded function. Then f **has a global maximum point at** x_0 **if**

$$f(x_0) \geq f(x) \qquad \text{for all } x \in X.$$

Similarly, the function f **has a global minimum point at** y_0 **if**

$$f(y_0) \leq f(x) \qquad \text{for all } x \in X.$$

The value of the function at a maximum point ,$f(x_0)$, is called **the maximum value** of the function and the value of the function $f(y_0)$, at a minimum point is called the **minimum value of the function**.

Remark 4.2.7.

1. The global maximum point (if there exists) it may be not unique, also the global maximum point (if there exists) it may be not unique. However the maximum value (if there exists) it is unique and the same thing for the minimum value. For example

$$f(x) = \sin x \qquad \text{for all } x \in \mathbb{R}.$$

has infinite number of global maximum points and global minimum points. In particular

$$f(\frac{\pi}{2} + 2n\pi) = 1 \qquad \text{for all } n \in \mathbb{N},$$

$$f(\frac{3\pi}{2} + 2n\pi) = -1 \qquad \text{for all } n \in \mathbb{N},$$

and

$$-1 \leq f(x) \leq 1 \qquad \text{for all } x \in \mathbb{R}.$$

2. It possible that a function f is bounded but does not has global maximum point (or global minimum point). For example

$$f(x) = 2x \qquad \text{for all } x \in (0,1)$$

note that for all $x \in (0,1)$, there exists x_1 such that

$$x < x_1 < 1 \qquad \text{and} \qquad f(x) < f(x_1).$$

Theorem 4.2.8. *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact space then there exists $x_0, y_0 \in X$ such that*

$$f(y_0) \leq f(x) \leq f(x_0), \qquad \text{for all } x \in X.$$

4.3 The Uniform Continuity

Definition 4.3.1. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called **uniformly continuous** on X if for every $\varepsilon > 0$ there exists $\delta > 0$ (δ depends on ε only) such that for every $x, y \in X$

$$\text{if} \quad d(x, y) < \delta \quad \Longrightarrow \text{then} \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Example 4.3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 7x - 3$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$, and let $\varepsilon > 0$. Then

$$d(x, y) = |x - y|$$

and

$$|f(x) - f(y)| = 7|x - y|$$

thus if we choose $\delta = \frac{\varepsilon}{7}$ then we get

$$d(x, y) = |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for every } x, y \in \mathbb{R}.$$

Hence f is uniformly continuous on $\mathbb{R}$.

Note that every uniformly continuous function is continuous, however the converse is not true. Consider the following example:

Example 4.3.3. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by

$$f(x) = \frac{1}{x}$$

Show that f is not uniformly continuous on $(0, \infty)$.

Solution: Let $\varepsilon = 1$ we will show that for every $\delta > 0$ there exists $x, y \in (0, \infty)$ such that

$$|x - y| < \delta \quad \text{but} \quad |f(x) - f(y)| > 1.$$

Let $\delta > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Suppose that $x, y \in (0, 1)$ such that $x = \frac{1}{n}$ and $y = \frac{2}{n}$, then

$$|x - y| < \frac{1}{n} \quad \text{but} \quad |f(x) - f(y)| = \frac{n}{2} > 1.$$

Hence f is not uniformly continuous.

Example 4.3.4. Let $a > 0$ and let $f : [a, \infty) \to \mathbb{R}$ be a function such that

$$f(x) = \frac{1}{x} \quad \text{for all } x \in [a, \infty).$$

Prove that f is uniformly continuous on $[a, \infty)$.
Solution: Let $\varepsilon > 0$ and $x, y \in [a, \infty)$. Then

$$d(x, y) = |x - y|$$

and

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{xy} < \frac{|y - x|}{a^2},$$

thus if we choose $\delta = a^2\varepsilon$ then we get

$$d(x, y) = |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in [a, \infty).$$

Therefore f is uniformly continuous on $[a, \infty)$.

Theorem 4.3.5. *Let (X, d) be a compact metric space. If $f : X \to \mathbb{R}$ is continuous function on X then f is uniformly continuous on X.*

Corollary 4.3.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then f is uniformly continuous on $[a, b]$.*